# STSCI 6940:

## Lecture 4

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$$\begin{cases} Y_{ij} = \sqrt{\frac{t\lambda}{n}}\, x_{0i} x_{0j} + W_{ij} \\ y_i = \sqrt{(1-t)\lambda q}\, x_{0i} + z_i \end{cases}$$

Posterior measure:

$$d\mathbb{P}(x \mid Y, y) \propto e^{\left(\sum_{i<j} \sqrt{\frac{\lambda t}{n}} W_{ij} x_i x_j + \frac{\lambda t}{n} x_i x_j x_{0i} x_{0j} - \frac{\lambda t}{2n} x_i^2 x_j^2 + \sum_{i=1}^{n} \sqrt{(1-t)\lambda q}\, z_i x_i - \frac{(1-t)\lambda q}{2} x_i^2 + (1-t)\lambda q x_i x_{0i}\right)} dP_0^n(x).$$

$$H_t(x) = \nearrow$$

$$\varphi(t) = \frac{1}{n} \mathbb{E} \log \int e^{H_t(x)} dP_0^n(x).$$

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$$F_n(\lambda) = \mathbb{E} \log \int e^{\sum_{i<j} \sqrt{\frac{\lambda}{n}} W_{ij} x_i x_j + \frac{\lambda}{n} x_i x_j x_{0i} x_{0j} - \frac{\lambda}{2n} x_i^2 x_j^2} dP_0^n(x).$$

$$F_n'(\lambda) = \sum_{i<j} \mathbb{E} \left\langle \frac{1}{2\sqrt{\lambda n}} W_{ij} x_i x_j + \frac{1}{n} x_i x_j x_{0i} x_{0j} - \frac{1}{2n} x_i^2 x_j^2 \right\rangle$$

First term:

$$\frac{1}{2\sqrt{\lambda n}} \mathbb{E}\left[ W_{ij} \langle x_i x_j \rangle \right].$$

$$= \frac{1}{2\sqrt{\lambda n}} \mathbb{E} \frac{d}{dW_{ij}} \left[ \frac{\int x_i x_j e^{H_n(x)} dP_0^n(x)}{\int e^{H_n(x)} dP_0^n(x)} \right]$$

$$= \frac{1}{2\sqrt{\lambda n}} \cdot \sqrt{\frac{\lambda}{n}} \left( \mathbb{E} \langle x_i^2 x_j^2 \rangle - \mathbb{E}\left[ \langle x_i x_j \rangle^2 \right] \right).$$

$$= \frac{1}{2n} \mathbb{E}\left( \langle x_i^2 x_j^2 \rangle - \langle x_i x_j \rangle^2 \right).$$

$$F_n'(\lambda) = \frac{1}{2n} \sum_{i<j} \mathbb{E}\left( \langle x_i^2 x_j^2 \rangle - \langle x_i x_j \rangle^2 \right)$$

$$+ \frac{1}{n} \sum_{i<j} \mathbb{E}\left[ \langle x_i x_j \rangle x_{0i} x_{0j} \right] - \frac{1}{2n} \sum_{i<j} \mathbb{E} \langle x_i^2 x_j^2 \rangle .$$

$$= -\frac{1}{2n} \sum_{i<j} \mathbb{E} \langle x_i x_j \rangle^2 + \frac{1}{n} \sum_{i<j} \mathbb{E} \langle x_i x_j \rangle x_{0i} x_{0j}$$

$$\mathbb{E}\left[ \langle x_i x_j \rangle x_{0i} x_{0j} \right] = \mathbb{E}\left[ \mathbb{E}[x_i x_j \mid Y] \cdot x_{0i} x_{0j} \right]$$

$$= \mathbb{E}\left[ \mathbb{E}[x_i x_j \mid Y]^2 \right]$$

$$= \mathbb{E}\left[ \langle x_i x_j \rangle^2 \right] .$$

$P_0$ has bounded support

$$F_n'(\lambda) = \frac{1}{2n} \sum_{i<j} \mathbb{E}\left[ \langle x_i x_j \rangle^2 \right] .$$

$$= \frac{1}{4n} \sum_{i,j=1}^{n} \mathbb{E}\left[ \langle x_i x_j \rangle^2 \right] - \frac{1}{4n} \sum_{i=1}^{n} \mathbb{E}\left[ \langle x_i^2 \rangle^2 \right].$$

$$= \frac{1}{4n} \sum_{i,j=1}^{n} \mathbb{E} \langle x_i^1 x_j^1 x_i^2 x_j^2 \rangle - O(1) ,$$

$$= \frac{n}{4} \mathbb{E} \langle \underbrace{\left( \frac{1}{n} \sum_{i=1}^{n} x_i^1 x_i^2 \right)}_{R_{1,2} = x^1 \cdot x^2 .}{}^2 \rangle - O(1) .$$

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$$\varphi'(t) = \frac{\lambda}{4} \mathbb{E} \langle R_{1,2}^2 \rangle - \lambda q \, \mathbb{E} \langle R_{1,2} \rangle - O\left(\frac{1}{n}\right).$$

$$= \frac{\lambda}{4} \underbrace{\mathbb{E} \langle (R_{1,2} - q)^2 \rangle}_{\geq 0} - \frac{\lambda q^2}{4} - O\left(\frac{1}{n}\right) .$$

$$\varphi'(t) \geq -\frac{\lambda q^2}{4} - o\left(\frac{1}{n}\right) . \quad \forall t, q$$

$$\varphi(1) - \varphi(0) \geq -\frac{\lambda q^2}{4} - o\left(\frac{1}{n}\right).$$

$$\varphi(1) = \frac{1}{n} F_n(\lambda)$$

$$\varphi(0) = \frac{1}{n} \mathbb{E} \log \int e^{\sum_{i=1}^{n} \sqrt{\lambda q}\, z_i x_i + \lambda q x_i x_{0i} - \frac{\lambda q}{2} x_i^2} dP_0^n(x).$$

$$= \mathbb{E}_{z, \varepsilon_0} \log \int e^{\sqrt{\lambda q}\, z \cdot \varepsilon + \lambda q \varepsilon \varepsilon_0 - \frac{\lambda q}{2} \varepsilon^2} dP_0(\varepsilon)$$

$$= \psi(\lambda q).$$

$$\Rightarrow \frac{1}{n} F_n(\lambda) \geq \psi(\lambda q) - \frac{\lambda q^2}{4} - o\left(\frac{1}{n}\right) \quad \forall q.$$

Lower bound

$$\boxed{\liminf_{n \to +\infty} \frac{F_n(\lambda)}{n} \geq \sup_{q \geq 0} \left\{ \psi(\lambda q) - \frac{\lambda q^2}{4} \right\} = \phi_{RS}(\lambda).}$$

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$x_1, \ldots, x_n$ iid

$$\log \mathbb{E}\, e^{\sum_{i=1}^{n} \lambda_i x_i} = n \log \mathbb{E}\, e^{\lambda x}$$

Upper bound ;

Several strategies :

1. Show that $\mathbb{E}\langle (R_{1,2} - \mathbb{E}\langle R_{1,2}\rangle_t)^2 \rangle_t \longrightarrow 0$

side information / pinning.

$$q_t = \mathbb{E}\langle R_{1,2} \rangle_t.$$

Adaptive interpolation method.

2. Cavity method :

$$\mathbb{E}\left[\langle (R_{1,2} - q)^2 \rangle\right] \leq \frac{c}{\sqrt{n}}.$$

3. use interpolation on a constrained version of the free energy where $x . x_0 = m$.
Somewhat simpler, difficult to generalize.

. Let's use strategy 3; $\qquad x \cdot x_0 = \frac{1}{n} \sum x_i x_{0i}$

Fix $m, x_0$

$$\varphi_n(m, x_0) = \frac{1}{n} \mathbb{E}_W \log \int \mathbb{1}\{x . x_0 = m\} e^{H(x)} \, dP_0^n(x).$$

Let's consider $P_0 = \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$ for simplicity

$T_n = \{ \frac{k}{n} : -n \leq k \leq n \}$ : set of possible values of $x . x_0$.

Fix $m \in T_n$.

$$--- \geq F_n(\lambda) \geq \mathbb{E}_{x_0} \varphi_n(m, x_0) \qquad \forall m.$$

laplace method $\displaystyle \int e^{n f(x)} dx = e^{n \max_x f(x) + o(n)}$.

Step 1: $\displaystyle \frac{1}{n} \log \int e^{n f(x)} dx = \max_x f(x) + o(1),$

Proposition:

$$\frac{1}{n} F_n(\lambda) \leq \mathbb{E}_{x_0} \max_{m \in T_n} \varphi_n(m, x_0) + O\left(\frac{\log n}{\sqrt{n}}\right).$$

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$$\cdot \ \frac{1}{n} F_n(\lambda) = \frac{1}{n} \mathbb{E} \log \sum_{m \in T_n} \int \mathbb{1}\{x . x_0 = m\} e^{H_n(x)} dP_0^n(x).$$

$$\leq \frac{1}{n} \mathbb{E} \log |T_n| \cdot \max_{m \in T_n} \int \underbrace{\mathbb{1}\{x . x_0 = m\} e^{H_n(x)}}\, dP_0^n(x)$$

$$(|T_n| = 2n+1) \rightsquigarrow \leq \frac{\log(2n+1)}{n} + \frac{1}{n} \mathbb{E}_{x_0, W} \max_{m \in T_n} \log \int \overset{\downarrow}{\cdot} \; dP_0^n(x).$$

$$X(m) = \frac{1}{n} \log \int \mathbb{1}\{x \cdot x_0 = m\} \, e^{H_n(x)} \, dP_0^n(x) .$$

want to show

$$\mathbb{E}_w \Big[ \max_{m \in T_n} X(m) \Big] \le \max_{m \in T_n} \mathbb{E}_w [X(m)] + o(1).$$

lemma: $\forall \gamma \geq 0$ ; $\forall m \in T_n$

$$\mathbb{E}_w \Big[ e^{\gamma (X(m) - \mathbb{E}_w [X(m)])} \Big]$$

$$\le \frac{K\gamma}{\sqrt{n}} e^{K\gamma^2/n}$$ for some $K \geq 0$

- $$\mathbb{E}_w \Big[ \max_{m \in T_n} ( X(m) - \mathbb{E}[X(m)] ) \Big]$$

$$= \frac{1}{\gamma} \mathbb{E}_w \Big[ \log e^{\gamma \max (X(m) - \mathbb{E}[X(m)])} \Big]$$

$$\le \frac{1}{\gamma} \log \mathbb{E}_w \Big[ \max_{m \in T_n} e^{\gamma (X(m) - \mathbb{E}[X(m)])} \Big]$$

$$\le \frac{1}{\gamma} \log \sum_{m \in T_n} \mathbb{E}_w \Big[ e^{\gamma (X(m) - \mathbb{E}_w [X(m)])} \Big]$$

$$\le \frac{1}{\gamma} \log \Big( (2n+1) \cdot \frac{K\gamma}{\sqrt{n}} e^{K\gamma^2/n} \Big) ,$$

$\gamma = \sqrt{n}$ $$\le \frac{\log(2n+1)}{\sqrt{n}} + \frac{e}{\sqrt{n}} = O\Big(\frac{\log n}{\sqrt{n}}\Big)$$

$$\mathbb{E}_w \Big[ \max_m X(m) \Big] - \max_m \mathbb{E}_w [X(m)]$$

$$\lesssim \mathbb{E}_w \max_m (X - \mathbb{E} X) ,$$

$$\frac{1}{n} F_n(\lambda) \leq \mathbb{E}_{x_0} \max_{m \in T_n} \mathbb{E}_w \frac{1}{n} \log \int \mathbb{1}\{x \cdot x_0 = m\} e^{H_n(x)} dP_0^n(x) + O\left(\frac{\log n}{\sqrt{n}}\right).$$

$$= \mathbb{E}_{x_0} \max_{m \in T_n} \varphi(m, x_0) + o(1).$$

- Step 2: upper bound $\varphi$ using Guerra's interpolation.

$$\varphi(m, x_0) = \frac{1}{n} \mathbb{E}_w \log \int \mathbb{1}\{x \cdot x_0 = m\} e^{H_n(x)} dP_0^n(x).$$

$$H_t(x) = \sum_{i<j} \sqrt{\frac{\lambda t}{n}} w_{ij} x_i x_j + \frac{\lambda t}{n} x_i x_j x_{0i} x_{0j} - \frac{\lambda t}{2n} x_i^2 x_j^2$$
$$+ \sum_{i=1}^{n} \sqrt{(1-t)\lambda q}\, z_i x_i + (1-t)\lambda m\, x_i x_{0i} - \frac{(1-t)q}{2} x_i^2$$

$$\varphi(t) = \frac{1}{n} \mathbb{E}_w \log \int \mathbb{1}\{x \cdot x_0 = m\} e^{H_t(x)} dP_0^n(x).$$

$$\varphi'(t) = -\frac{\lambda}{2n^2} \sum_{i<j} \mathbb{E}\langle x_i x_j \rangle^2 + \frac{\lambda}{n^2} \sum_{i<j} \mathbb{E}\langle x_i x_j \rangle x_{0i} x_{0j}$$
$$+ \frac{\lambda q}{n} \sum_{i=1}^{n} \mathbb{E}[\langle x_i \rangle^2] - \frac{\lambda m}{n} \sum_{i=1}^{n} \mathbb{E}[\langle x_i \rangle x_{0i}]$$
$$= -\frac{\lambda}{4} \mathbb{E}\langle (R_{1,2} - q)^2 \rangle + \frac{\lambda}{2} \mathbb{E}\langle \underbrace{(R_{10} - m)^2}_{=0} \rangle$$
$$+ \frac{\lambda}{4} q^2 - \frac{\lambda}{2} m^2 + o\left(\frac{1}{n}\right).$$

$$\langle \cdot \rangle = \frac{\int \cdot\, \mathbb{1}\{R_{10} = m\} e^{\_\_\_}}{\int \mathbb{1}\{R_{10} = m\} e^{\_\_\_}}.$$

$$\varphi'(t) \leq \frac{\lambda}{4} q^2 - \frac{\lambda}{2} m^2 + o\left(\frac{1}{n}\right).$$

$$\varphi(1) - \varphi(0) \leq \frac{\lambda}{4} q^2 - \frac{\lambda}{2} m^2 .$$

$$\varphi(1) = \varphi(m, x_0)$$

$$\varphi(0) = \frac{1}{n} \mathbb{E}_w \log \int \mathbb{1}\{x \cdot x_0 = m\} e^{\sum_{i=1}^{n} \sqrt{\lambda q}\, z_i x_i + \lambda m\, x_i x_{0i} - \frac{\lambda q}{2} x_i^2} dP_0^n(x).$$

$$\leq \frac{1}{n} \mathbb{E} \log \int e^{\sum \text{———}} dP_0^n(x) .$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_z \log \int e^{\lambda q z \varepsilon + \lambda m \varepsilon x_{0i} - \frac{\lambda q}{2} \varepsilon^2} dP_0(\varepsilon) .$$

$$= \hat{F}(x_0, \lambda q, \lambda m) .$$

we proved that

proposition :

$$\boxed{\varphi(m, x_0) \leq \hat{F}(x_0, \lambda q, \lambda m) + \frac{\lambda q^2}{4} - \frac{\lambda m^2}{2} + O\left(\frac{1}{n}\right)}$$