STSCI 6940:

Lecture 18 Date: 04/15

- Low degree likelihood ratio method.

→ Predicting computational difficulty of binary hypothesis testing.

Distinguish between two distributions $\mathbb{P}$, $\mathbb{Q}$ on $\mathbb{R}^n$

$H_1 : Y \sim \mathbb{P}_n$, $H_0 : Y \sim \mathbb{Q}_n$.

Test minimizes Type I + Type II error is

Likelihood ratio test $L_n = \frac{d\mathbb{P}_n}{d\mathbb{Q}_n}$ $(\geq 1$ or $\leq 1)$

Restrict theory to the class of polynomial tests.

→ $p : \mathbb{R}_{\leq d}[Y] \to \mathbb{R}$

→ Apply test $p(Y)$ $(\geq \tau$ or $\leq \tau)$.

- Another characterization the LR :

Let's assume that $\mathbb{P}_n \ll \mathbb{Q}_n$, and

$\mathbb{Q}_n$ has all moments finite.

$Y_i^k \in \mathbb{L}^1(\mathbb{Q})$. $\forall i, \forall k$

• $$\max_{f \in \mathbb{L}^2(\mathbb{Q})} \ \mathbb{E}_{\mathbb{P}}[f(Y)] = \langle L, f \rangle_{\mathbb{L}^2(\mathbb{Q})}.$$

$$\text{s.t.} \quad \mathbb{E}_{\mathbb{Q}}[f(Y)^2] = 1 .$$

$$\| L \|_2^2 = 1$$

$\hookrightarrow$ argmax is $f_* = \dfrac{L}{\|L\|_2}$

(2-norm is in $\mathbb{L}^2(\mathbb{Q})$)

$$\|L\|_2 = \mathbb{E}_{\mathbb{Q}}[L^2]^{\frac{1}{2}} .$$

objective : $\|L\|_2$.

Main idea (Hopkins, Steurer 16 (?)).

restrict optimization to bounded degree polynomials .

$$\rightarrow \quad \max_{f \in \mathbb{R}_{\leq d}[Y]} \ \mathbb{E}_{\mathbb{P}}[f(Y)] .$$

$$\text{s.t.} \quad \mathbb{E}_{\mathbb{Q}}[f(Y)^2] = 1$$

$$\mathbb{E}_{\mathbb{P}}[f(Y)] = \mathbb{E}_{\mathbb{Q}}[L(Y) f(Y)]$$

$$= \mathbb{E}_{\mathbb{Q}}[\mathrm{Proj}_{\leq d}(L) \cdot f]$$

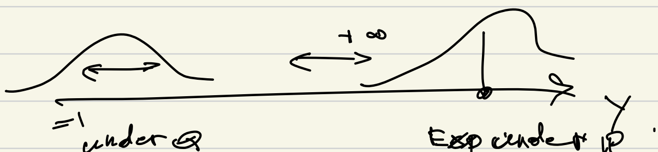
$\Rightarrow$ maximizer is $\quad f_* = \dfrac{L_{\leq d}}{\|L_{\leq d}\|_2}$.

Can (in principle) compute $L_{\leq d}$ in $O(n^d)$ time.

$\rightarrow$ objective : $\|L_{\leq d}\|_2$.

Prediction on computational difficulty:

whether $\|L_{\leq d}\|_2$ is bounded or diverges.



Analogy with contiguity via second moment:

$$\mathbb{P}_n \triangleleft \mathbb{Q}_n \quad \text{if } \overline{\lim_n}\, \mathbb{E}[L_n^2] < +\infty$$

Application: Additive Gaussian channel.

$H_1 : Y = X + Z$, $\quad Z \sim N(0, I_n)$,
$X \sim P_0$ on $\mathbb{R}^n$.

$H_0 : Y = Z$.

LR: $\quad L(Y) = \mathbb{E}_{X \sim P_0}\left[e^{\langle Y, X\rangle - \frac{\|X\|^2}{2}} \right]$.

How to compute $L_{\leq d}$?

- Consider the basis of Hermite polynomials.

Hilbert space is $L^2(\mathcal{Q}_n)$. $\mathcal{Q}_n = N(0, I_n)$

(univariate)

Hermite polynomials: $h_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}\left(e^{-x^2/2}\right)$

$k \geq 0,\ x \in \mathbb{R}$

$$\mathbb{E}_{X \sim N(0,1)} \left[h_k(x)\, h_{k'}(x) \right] = k!\ \delta_{k,k'}$$

$$\hat{h}_k(x) = \frac{h_k(x)}{\sqrt{k!}}$$ form an orthobasis for $L^2(N(0,1))$.

- Multivariate Hermite polynomials:

$x \in \mathbb{R}^n$

$\alpha \in \mathbb{N}^n$

$$H_\alpha(x) := \prod_{i=1}^{n} h_{\alpha_i}(x_i)$$

$$\hat{H}_\alpha(x) := \prod_{i=1}^{n} \hat{h}_{\alpha_i}(x_i).$$

$\Rightarrow$ orthobasis for $L^2(\mathcal{Q}_n)$.

$$\Rightarrow \left\{ \begin{aligned} L(Y) &= \sum_{\alpha \in \mathbb{N}^n} \langle L, \hat{H}_\alpha \rangle \cdot \hat{H}_\alpha(Y). \\ L_{\leq d}(Y) &= \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} \langle L, \hat{H}_\alpha \rangle \cdot \hat{H}_\alpha(Y). \end{aligned} \right.$$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

- Compute inner products: $\langle L, \hat{H}_\alpha \rangle$.

- Lemma 1: $\| L_{\leq d} \|_2^2 = \mathbb{E}_{x_1, x_2 \sim P} \left[\sum_{k=0}^{d} \frac{1}{k!} \langle x_1, x_2 \rangle^k \right]$

Parallel with $\| L \|_2^2$:

$$\| L \|_2^2 = \mathbb{E}_Q [L^2] = \mathbb{E}_{x_1, x_2 \sim P} \left[e^{\langle x_1, x_2 \rangle} \right]$$

pf: $\mathbb{E}_Q [L^2] = \mathbb{E}_P [L]$.

$$= \mathbb{E}_P \left[\mathbb{E}_X \left[e^{\langle Y, X \rangle - \frac{\|X\|^2}{2}} \right] \right]$$

$$= \mathbb{E}_X \left[\mathbb{E}_{X_0, Z} \left[e^{\langle X_0 + Z, X \rangle - \|X\|^2/2} \right] \right]$$

$$= \mathbb{E}_X \, \mathbb{E}_{X_0} \left[e^{\langle X_0, X \rangle} \cdot \underbrace{\mathbb{E}_Z \left[e^{\langle Z, X \rangle - \|X\|^2/2} \right]}_{=1} \right]$$

$$= \mathbb{E}_{X_1 X_2} \left[e^{\langle X_1, X_2 \rangle} \right].$$

$$= \mathbb{E}_{X_1, X_2} \left[\sum_{k=0}^{+\infty} \frac{1}{k!} \langle X_1, X_2 \rangle^k \right]$$

Proof of Lemma 1:

Gaussian integration by parts.

$$\mathbb{E}_{z \sim N(0,1)} [z f(z)] = \mathbb{E} [f'(z)]$$

$\forall f$ weakly diff, poly growth at $\pm\infty$.

A generalization of G.I.P :

$$\mathbb{E}_{z\sim N(0,1)}\left[h_k(z)\, f(z) \right] = \mathbb{E}\left[f^{(k)}(z) \right].$$

for f w-diff, poly growth.

- $$L_{\le d} = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le d}} \langle L, \hat{H}_\alpha \rangle \cdot \hat{H}_\alpha$$

$$\| L_{\le d} \|_2^2 = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le d}} \langle L, \hat{H}_\alpha \rangle^2 .$$

$$\rightarrow \langle L, \hat{H}_\alpha \rangle = \mathbb{E}_{Q}\left[L \cdot \hat{H}_\alpha \right].$$

$$= \mathbb{E}_{Q}\left[\prod_{i=1}^{n} h_{\alpha_i}(y_i) \cdot L(y) \right] \frac{1}{\prod_{i=1}^{n} \sqrt{\alpha_i!}} .$$

$$= \mathbb{E}_{Q}\left[\underbrace{\frac{\partial^\alpha}{\partial y^\alpha} L(y)} \right] \frac{1}{\prod_{i=1}^{n} \sqrt{\alpha_i!}} .$$

$$= \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} L .$$

$$L = \mathbb{E}_{x\sim P_0}\left[e^{\langle y, x\rangle - \|x\|^2/2} \right].$$

$$\frac{\partial^{\alpha_i} L}{\partial y_i^{\alpha_i}} = \mathbb{E}_{x\sim P_0}\left[x_i^{\alpha_i}\, e^{\overline{\qquad}} \right].$$

$$\langle L, \hat{H}_\alpha \rangle = \frac{1}{\sqrt{\prod_{i=1}^n \alpha_i !}} \cdot \mathbb{E}_{Y\sim \mathbb{R}} \mathbb{E}_{X\sim P_0} \left[\prod_{i=1}^n x_i^{\alpha_i} e^{\langle Y, X\rangle - \frac{\|X\|^2}{2}} \right]$$

$$= \underline{\qquad} \cdot \mathbb{E}_{X\sim P_0} \left[\prod_{i=1}^n x_i^{\alpha_i} \right].$$

$$\Rightarrow \| L_{\leq d} \|_2^2 = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} \frac{1}{\prod_{i=1}^n \alpha_i !} \mathbb{E}_{X, X' \sim P_0} \left[\prod_{i=1}^n (X_i X_i')^{\alpha_i} \right].$$

$$= \sum_{k=0}^d \frac{1}{k!} \mathbb{E}_{X, X' \sim P_0} \left[\sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \underbrace{\frac{k!}{\prod_{i=1}^n \alpha_i !}}_{\binom{k}{\alpha_1, \ldots, \alpha_n}} \prod_{i=1}^n (X_i X_i')^{\alpha_i} \right].$$

$$= \sum_{k=0}^d \frac{1}{k!} \mathbb{E}_{X, X' \sim P_0} \left[\left(\sum_{i=1}^n X_i X_i' \right)^k \right].$$

$$= \sum_{k=0}^d \frac{1}{k!} \mathbb{E}_{X, X' \sim P_0} \left[\langle X, X' \rangle^k \right].$$

■

Another view : Fix $x \in \mathbb{R}$

$y \mapsto e^{xy - x^2/2}$ expanded on the basis of $(h_k)_{k \geq 0}$?

$$e^{xy - x^2/2} = \sum_{k=0}^{+\infty} \frac{1}{k!} x^k h_k(y). \qquad \forall y \in \mathbb{R}.$$

$$\Rightarrow \quad L = \mathbb{E}_{X \sim P_0} \left[e^{\langle Y, X \rangle - \|X\|^2/2} \right]$$

$$= \mathbb{E}_{X \sim P_0} \left[\prod_{i=1}^{n} \left(\sum_{k=0}^{+\infty} \frac{1}{k!} x_i^k \, h_k(y_i) \right) \right].$$

$$= \sum_{\alpha \in \mathbb{N}^n} \mathbb{E}_{X \sim P_0} \left[\prod_{i=1}^{n} \frac{x_i^{\alpha_i}}{\alpha_i!} h_{\alpha_i}(y_i) \right].$$

$$= \sum_{\alpha \in \mathbb{N}^n} \frac{\mathbb{E}_{X \sim P_0} \left[\prod_{i=1}^{n} x_i^{\alpha_i} \right]}{\sqrt{\prod_{i=1}^{n} \alpha_i!}} \hat{H}_\alpha(Y).$$

$$\Rightarrow \quad L_{\leq d} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} .$$

$$\Rightarrow \quad \mathbb{E}\left[L_{\leq d}^2 \right] = \sum_{k=0}^{d} \frac{1}{k!} \mathbb{E}_{X, X' \sim P_0} \left[\langle X, X' \rangle^k \right].$$

Special case spiked wigner model :

$$Y_{ij} = \sqrt{\frac{\lambda}{n}} \, x_i x_j + W_{ij} \qquad W_{ij} \sim N(0,1) \quad i < j$$

$x_i \sim P_0$ zero mean, unit variance.

λ_c : rea threshold $(\lambda_c < 1)$.

Look at the LDLR :

$$\| L_{\leq d} \|_2^2 = \sum_{k=0}^{d} \left(\frac{\lambda}{2n} \right)^k \frac{1}{k!} \mathbb{E}_{X, X' \sim P_0} \left[\langle X, X' \rangle^{2k} \right].$$

$$= \sum_{k=0}^{d} \frac{1}{2^k} \frac{\lambda^k}{k!} \mathbb{E}_{X, X' \sim P_0} \left[\left(\frac{\langle X, X' \rangle}{\sqrt{n}} \right)^{2k} \right].$$

By the CLT:

$$\lim_{n \to +\infty} \| L_{\leq d} \|_2^2 = \sum_{k=0}^{d} \frac{1}{2^k} \frac{\lambda^k}{k!} \mathbb{E}[g^{2k}].$$

Fact: If $\left|\frac{a_{k+1}}{a_k}\right| \to r > 1$ then $\sum |a_k| = +\infty$.

< 1 — — $< +\infty$

$$\mathbb{E}[g^{2k}] = (2k-1)(2k-3) \cdots 1.$$

$$a_k = \frac{1}{2^k} \frac{\lambda^k}{k!} \mathbb{E}[g^{2k}].$$

$$\frac{a_{k+1}}{a_k} = \frac{1}{2} \frac{\lambda}{k+1} \frac{(2k+1)(2k-1)(2k-3) \cdots}{(2k-1)(2k-3) \cdots}$$

$$= \frac{1}{2} \frac{\lambda}{k+1} \cdot (2k+1) \longrightarrow \lambda.$$

If $\lambda < 1$: $\sup_{d \geq 0} \lim_{n \to +\infty} \| L_{\leq d} \|_2^2 < +\infty$.

$\Rightarrow$ LDLR predicts testing is computationally difficult for all $\lambda < 1$.