STSCI 6940:

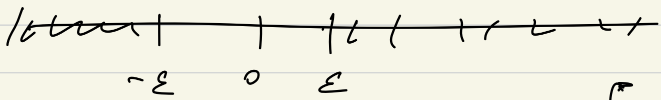
Lecture 17 — Date: 04/08

Concentration of overlaps:

$$\mathbb{E}\langle \mu\{ |R_{1,0}| \geq \varepsilon \}\rangle \leq 2 e^{-c(\varepsilon) n} \quad \forall \varepsilon > 0.$$

- $\mathbb{E}\langle \mu\{ R_{1,0} = q \}\rangle \leq 2 e^{-c(\varepsilon) n}$

$$\forall \ |q| \geq \varepsilon .$$



$$\mathbb{E}\langle \mu\{ R_{1,0} = q \}\rangle = \mathbb{E}\left[\frac{\int \mathbb{1}\{R_{1,0}=q\}\, e^{H(x)}\, dP_0^n(x)}{\int e^{H(x)}\, dP_0^n(x)} \right]$$

$$= \mathbb{E} \exp\left(n \cdot \left(\frac{1}{n} \log \int \mathbb{1}\{R_{1,0}=q\}\, e^{H(x)}\, dP_0^n(x) - \frac{1}{n} \log \int e^{H(x)}\, dP_0^n(x) \right)\right).$$

$$f_n(q) = \frac{1}{n} \log \int \mathbb{1}\{R_{1,0}=q\}\, e^{H(x)}\, dP_0^n(x)$$

$$f_n = \frac{1}{n} \log \int e^{H(x)}\, dP_0^n(x).$$

Proposition 1: $\forall\ q \in \mathbb{R}$, $\forall\ u \geq 0$.

$$f_n(q) - f_n \leq F(\lambda, |q|) - \sup_{q \geq 0} F(\lambda, q) + u + \frac{K}{n} \quad \text{w.p.} \geq 1 - e^{-nu^2/K}.$$

$$F(\lambda, q) = \psi(\lambda q) - \frac{\lambda q^2}{4}.$$

$$\psi(r) = \mathbb{E}_{z, x_0} \int e^{\sqrt{r} z x + r x x_0 - \frac{r}{2} x^2} dP_0(x)$$

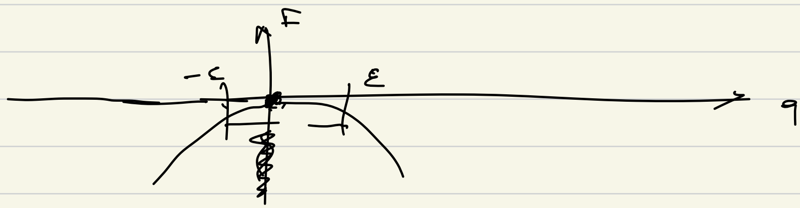
we proved :

$$\mathbb{E} f_n \longrightarrow \phi_{RS}(\lambda) = \sup_{q \geq 0} F(\lambda, q).$$

For $\lambda < \lambda_c$: $q = 0$ is unique maximizer of $q \to F(\lambda, q)$.

$$F(\lambda, 0) = 0 .$$

$\forall |q| \geq \varepsilon :$ $F(\lambda, q) - 0 \leq -c(\varepsilon)$.



$$\Rightarrow f_n(q) - f_n \leq -c(\varepsilon) + u + \frac{K}{n}$$

$$\text{w.p} \geq 1 - e^{-nu^2/K} .$$

Let's take $u = + \frac{c(\varepsilon)}{2}$

$$\Rightarrow \mathbb{E} \langle \mathbb{1}\{R_{10} = q\} \rangle = \mathbb{E} \exp(n(f_n(q) - f_n)).$$

$$\leq \exp\left(n\left(-\frac{c(\varepsilon)}{2} + \frac{K}{n}\right)\right) + e^{-nu^2/K}.$$

$$u = \frac{c(\varepsilon)}{2}.$$

$$\leq 2 e^{-c'(\varepsilon) n}.$$

Proof of Proposition 1:

Using Guerra's interpolation:

$$\begin{cases} \mathbb{E} f_n(\lambda) \geq \sup_{q \geq 0} F(\lambda, q) - \frac{K}{n}. \\ \mathbb{E}_w f_n(q) \leq \hat{F}(\lambda, |q|, q) + \frac{K}{n}. \end{cases}$$

$$\hat{F}(\lambda, m, q) = \frac{1}{n} \sum_{i=1}^{n} \hat{\varphi}(\lambda q, \lambda m x_{0i}) - \frac{\lambda}{2} m^2 + \frac{\lambda}{4} q^2.$$

$$\hat{\varphi}(r, s) = \mathbb{E}_{\xi} \log \int e^{\sqrt{r}\xi + sx - \frac{r}{2} x^2} dP_0(x).$$

Show :

1. $f_n - \mathbb{E}_w f_n$ concentrates. (Gaussian concentration of Lipschitz functions)
2. $\mathbb{E}_w f_n - \mathbb{E}_{w,x_0} f_n$ concentrates (concentration of lip. conv functions w.r.t iid bounded r.v)
3. $f_n(q) - \mathbb{E}_w f_n(q)$ concentrates. (concentration w.r.t Gaussian)
4. $\hat{F} - \mathbb{E}_{x_0} \hat{F}$ concentrates. (Hoeffding)

$$\Rightarrow f_n(q) - f_n = \overbrace{\left(f_n(q) - \mathbb{E}_w f_n(q)\right)}^{A_1}$$

$$+ \mathbb{E}_w f_n(q) - \mathbb{E}_w f_n - \overbrace{\left(f_n - \mathbb{E}_w f_n\right)}^{A_2}$$

$$\leq A_1 + A_2 + \hat{F}(\lambda, |q|, q) - \mathbb{E}_w f_n$$

$$= A_1 + A_2 + \underbrace{\hat{F} - \mathbb{E}\hat{F}}_{A_3} + \mathbb{E}\hat{F} - \mathbb{E} f_n$$

$$+ \underbrace{\mathbb{E}_w f_n - \mathbb{E} f_n}_{A_4}$$

$$\mathbb{P}(|A_i| \geq t) \leq e^{-nt^2/K} \quad \forall\ i = 1,2,3,4.$$

$$\mathbb{E}\hat{F}(\lambda, |q|, q) = \mathbb{E}\,\psi(\lambda|q|, \lambda q) - \frac{\lambda q^2}{4}$$

- Lemma: $\psi(r, -s) \leq \psi(r, s) \quad \forall\ s \geq 0.$

$$\psi(r,s) = \mathbb{E}_{z, x_0} \log \int e^{\sqrt{r} z x + s x x_0 - \frac{r}{2} x^2} dP_0(x).$$

$$\leq \mathbb{E}\,\psi(\lambda|q|, \lambda|q|) - \frac{\lambda |q|^2}{4}$$

$$= F(\lambda|q|).$$

we already know $\mathbb{E} f_n \geq \sup_{q \geq 0} F(\lambda q)$.

$$\Rightarrow \quad f_n(q) - f_n \leq F(\lambda|q|) - \sup_{q \geq 0} F(\lambda, q) + u$$

with prob $1 - 4e^{-nu^2/K(\lambda)}$.

$$f_n(q) - f_n \leq F(\lambda, |q|) - \sup_{q \geq 0} F(\lambda, q).$$

This closes the sketch of proof of

$\forall \lambda < \lambda_c$: $\log L \rightsquigarrow \mathcal{N}(\mu, \sigma^2)$. under $\mathbb{P}_\lambda$

$$\mu = \frac{1}{4}\left(-\log(1-\lambda) - \lambda\right) \qquad ①$$

$$\sigma^2 = 2\mu$$

① $\Rightarrow$ $\forall \lambda < \lambda_c$: $\log L \rightsquigarrow N(-\mu, \sigma^2)$ under $\mathbb{P}_0$.

- Portmanteau characterization of convergence in distribution:

$\forall f \geq 0$ continuous

$$\underline{\lim}\ \mathbb{E}_{\mathbb{P}_\lambda}[f(\log L)] \geq \mathbb{E}[f(Z)]$$

$$Z \sim N(\mu, \sigma^2).$$

$$\mathbb{E}_{\mathbb{P}_0}[f(\log L)] = \mathbb{E}_{\mathbb{P}_\lambda}\left[\frac{d\mathbb{P}_0}{d\mathbb{P}_\lambda} \cdot f(\log L)\right]$$

$$= \mathbb{E}_{\mathbb{P}_\lambda}\left[e^{-\log L} f(\log L)\right].$$

$$g(x) = e^{-x} f(x) \geq 0$$

Portmanteau:

$$\Rightarrow \underline{\lim}\ \mathbb{E}_{\mathbb{P}_0}[f(\log L)] \geq \mathbb{E}[e^{-Z} f(Z)]$$

$$Z \sim N(\mu, \sigma^2).$$

$$\mathbb{E}[e^{-Z} f(Z)] = \int e^{-z} e^{-\frac{(z-\mu)^2}{2\sigma^2}} f(z) \frac{dz}{\sqrt{2\pi}}$$

$$-z - \frac{(z-\mu)^2}{2\sigma^2} = -z - \frac{z^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} + \frac{\mu}{\sigma^2} z$$

$\sigma^2 = 2\mu$: $\quad -z + \frac{\mu}{\sigma^2} z = -z + \frac{1}{2} z = -\frac{1}{2} z$

$$= -\frac{\mu}{\sigma^2} z .$$

$$\Rightarrow \quad -z - \frac{(z-\mu)^2}{2\sigma^2} = -\frac{(z+\mu)^2}{2\sigma^2} .$$

$$\Rightarrow \quad \mathbb{E}\left[e^{-Z} f(Z)\right] = \int e^{-\frac{(z+\mu)^2}{2\sigma^2}} f(z) \frac{dz}{\sqrt{2\pi}}$$

$$= \mathbb{E} f(Z)$$

where $Z \sim N(-\mu, \sigma^2)$.

$\Rightarrow \quad \log L \rightsquigarrow N(-\mu, \sigma^2)$ under $\mathbb{P}_0$.

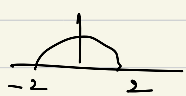
Efficient tests:

$$Y = \sqrt{\frac{\lambda}{n}}\, x x^T + W . \qquad \|x\|_2 = 1$$

$\mu_1 \geq \dots \geq \mu_n$. eigenvals of Y.

$$T_n = \sum_{i=1}^{n} \phi(\mu_i) . \qquad \phi : \mathbb{R} \to \mathbb{R}.$$

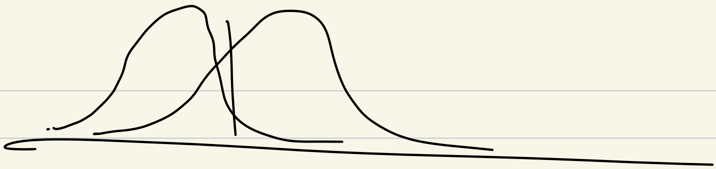
for $\lambda < 1$ $\quad \frac{1}{n} T_n \longrightarrow \int_{-2}^{2} \phi(x) \frac{\sqrt{4-x^2}}{2\pi} dx$



Fluctuations:

$$T_n - n \int_{-2}^{2} \phi(x) \frac{\sqrt{4-x^2}}{2\pi} dx \rightsquigarrow N(m_\phi, v_\phi) .$$

eigenvalue rigidity.



V_ϕ does not depend on λ.

choose ϕ to maximize

$$\left| \frac{m_\phi(\lambda) - m_\phi(\lambda=0)}{\sqrt{V_\phi(\lambda=0)}} \right|$$

Theorem [Chung-Lee 2019]:

maximizers are of the form:

$$\phi(x) = \log \frac{1}{1-\sqrt{\lambda}\, x + \lambda} + \sqrt{\lambda}\left(\frac{2}{\sigma^2} - 1\right) x$$

$\sigma^2 = \mathrm{var}(w_{ii})$.

with choice and $\sigma^2 = +\infty$

Test: reject null if $T_n \geq \frac{m_\phi(\lambda)+m_\phi(\lambda=0)}{2}$

$$\mathrm{err}_n(\text{Test}) \longrightarrow \mathrm{err}(LLR) = \mathrm{erfc}\left(\frac{\sqrt{\mu}}{2}\right).$$

$\forall \lambda < 1$.