STSCI 6940:

Lecture 16 Date: 04/06

Let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be function of $x^0, \dots, x^k$

$$\left| \frac{d}{dt} \mathbb{E} \langle f(x^0, \dots, x^k) \rangle_t \right|$$

$$\leq \sum_{a,b} |c_{a,b}| \, \mathbb{E} \langle |x_n^{(a)} x_n^{(b)}| \cdot |R^-_{a,b}| \cdot |f(\quad)| \rangle_t$$

$$\leq K \cdot \left[\mathbb{E} \langle |R^-_{a,b}|^{\tau_1} \rangle \right]^{1/\tau_1}$$

$$\cdot \, \mathbb{E} \left[\langle |f|^{\tau_2} \rangle \right]^{1/\tau_2} \quad \text{(Hölder)}.$$

$$\forall \, \tau_1 + \tau_2 = 1.$$

In particular

$$\left| \frac{d}{dt} \mathbb{E} \langle f \rangle_t \right| \leq K \cdot \mathbb{E} \langle |f| \rangle_t \quad \forall t$$

$$f \geq 0 \qquad = K \, \mathbb{E} \langle f \rangle_t \quad \forall t.$$

$\Rightarrow$ Gronwall: $\mathbb{E} \langle f \rangle_t \leq K \cdot \mathbb{E} \langle f \rangle_0$

- $y' \leq a y \qquad y(t) \leq y(0) \, e^{at}$

The second statement: $\forall \lambda < 1$

$$\mathbb{E} \left[\langle x_n^2 x_{0n}^2 \rangle e^{is \log L} \right]$$

$$= \mathbb{E} \left[e^{is \log L} \right] + \delta$$

$$|\delta| \leq K \cdot n \, \mathbb{E} \langle |R_{1,0}|^3 \rangle .$$

$$\varphi(t) = \mathbb{E}\left[\langle x_n^2 x_{on}^2 \rangle_t \; X(t) \right].$$

$$\varphi(0) = \mathbb{E}\left[x_n^2 x_{on}^2 \right] \cdot \mathbb{E}\left[X(0) \right].$$

$x_n, x_{on} \sim P_0$ indep

$$= \mathbb{E}\left[X(0) \right].$$

$$= \mathbb{E}\left[X(1) \right] + \delta \quad , \quad |\delta| \le \mathbb{E} \langle |R_{1,0}| \rangle.$$

$$\varphi'(t) = \sum_{a,b} c_{ab} \mathbb{E}\left[\langle x_n^2 x_{on}^2 \; x_n^{(a)} x_n^{(b)} R^-_{a,b} \rangle_t X(t) \right]$$

$$|\varphi'(t)| \le K \cdot \mathbb{E} \langle |R^-_{1,0}| \rangle$$

$$\le K \, \mathbb{E} \langle |R_{1,0}| \rangle + \frac{K}{n}.$$

$$\le \frac{K}{\sqrt{n}}$$

$$\Rightarrow |\varphi(1) - \varphi(0)| \le \frac{K}{\sqrt{n}}$$

$$|\varphi(0) - \mathbb{E} e^{is\log L}| \le \frac{K}{\sqrt{n}}.$$

$$\Rightarrow \mathbb{E}\left[\langle x_n^2 x_{on}^2 \rangle e^{is\log L} \right] = \mathbb{E}\left[e^{is\log L} \right] + \delta$$

$$|\delta| \le K \mathbb{E} \langle |R_{1,0}| \rangle \le \frac{K}{\sqrt{n}}.$$

$(\forall \lambda < 1)$ $(\forall \lambda < \lambda_c)$

$\blacksquare$

Convergence of overlaps:

Theorem : $\forall \lambda < \lambda_c$:

$$\mathbb{E} \langle R_{1,0}^4 \rangle \leq \frac{K}{n^2}.$$

Two main ingredients: cavity + an apriori bound on $R_{1,0}$.

(Talagrand 2005)

start by proving $\mathbb{E} \langle R_{1,0}^2 \rangle \leq \frac{K}{n}$.

use cavity method.

$$\mathbb{E} \langle R_{1,0}^2 \rangle = \mathbb{E} \langle \left(\frac{1}{n} \sum_{i=1}^{n} x_i x_{0i}\right) R_{1,0} \rangle$$

$$= \mathbb{E} \langle x_n x_{0n} \cdot R_{1,0} \rangle.$$

$$= \frac{1}{n} \mathbb{E} \langle x_n^2 x_{0n}^2 \rangle + \mathbb{E} \langle x_n x_{0n} R_{1,0}^- \rangle.$$

$\Rightarrow$ Interpolate the last variable.

$$\left| \mathbb{E} \langle x_n^2 x_{0n}^2 \rangle_t - 1 \right| \leq K \cdot \mathbb{E} \langle |R_{1,0}| \rangle + \frac{K}{n}.$$

$$\mathbb{E} \langle x_n x_{0n} R_{1,0}^- \rangle = \lambda \mathbb{E} \langle (R_{1,0}^-)^2 \rangle + \delta$$

$$|\delta| \leq K \cdot \mathbb{E} \langle |R_{1,0}^3| \rangle.$$

$$\mathbb{E}\langle x_n x_{on} R_{1,0}^- \rangle = \lambda\, \mathbb{E}\langle (R_{1,0})^2 \rangle + \delta$$

$$|\delta| \leq K\, \mathbb{E}\langle |R_{1,0}^3| \rangle + \frac{K}{n} \mathbb{E}\langle |R_{1,0}| \rangle + \frac{K}{n^2}.$$

$$\mathbb{E}\langle R_{1,0}^2 \rangle = \frac{1}{n} + \lambda\, \mathbb{E}\langle R_{1,0}^2 \rangle + \delta.$$

need to show apriori bound on $R_{1,0}$:

Theorem: $\forall \lambda < \lambda_c$, $\forall \varepsilon > 0$

$$\mathbb{E}\langle \mathbb{1}\{ |R_{1,0}| \geq \varepsilon \} \rangle \leq e^{-c(\varepsilon) n}.$$

$c(\varepsilon) > 0$,

(convergence of $R_{1,0}$ in probability).

Fix $\varepsilon > 0$:

$$\Rightarrow \cdot\ \mathbb{E}\langle |R_{1,0}| \rangle \leq \varepsilon + \mathbb{E}\langle \underbrace{|R_{1,0}|}_{\leq K} \mathbb{1}\{|R_{1,0}| \geq \varepsilon\} \rangle$$

$$\leq \varepsilon + K e^{-cn}.$$

$$\cdot\ \mathbb{E}\langle |R_{1,0}|^3 \rangle = \mathbb{E}\langle |R_{1,0}| \cdot (R_{1,0})^2 \rangle.$$

$$\leq \varepsilon\, \mathbb{E}\langle R_{1,0}^2 \rangle + K \cdot e^{-cn}.$$

Putting things together:

$$\mathbb{E}\langle R_{1,0}^2 \rangle \leq \frac{1}{n} + \lambda\, \mathbb{E}\langle R_{1,0}^2 \rangle +$$

$$K.\varepsilon\, \mathbb{E}\langle R_{1,0}^2 \rangle + K e^{-cn} +$$

$$\frac{K}{n}(\varepsilon + K e^{-cn}) + \frac{c}{n^2}.$$

$$(1-\lambda-K\varepsilon)\,\mathbb{E}\langle R_{1,0}^2\rangle \leq \frac{1+K\varepsilon}{n} + Ke^{-cn} + \frac{1}{n^2}$$

if $1-\lambda-K\varepsilon > 0$ wich is true for ε small enough

we get $\mathbb{E}\langle R_{1,0}^2\rangle \leq \frac{K}{n}$ $\forall \lambda < 1$.

- want to go from second moment to fourth moment.

$$\mathbb{E}\langle R_{1,0}^4\rangle = \mathbb{E}\left\langle \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_{i0}\right)\left(\frac{1}{n}x_n x_{0n} + R_{1,0}^-\right)^3\right\rangle$$

$$= \mathbb{E}\langle x_n x_{0n} (\qquad\qquad)^3 \rangle$$

$$= \mathbb{E}\langle x_n x_{0n} (R_{1,0}^-)^3\rangle + \frac{3}{n}\mathbb{E}\langle (R_{1,0}^-)^2 (x_n x_{0n})^2\rangle$$

$$+ \frac{3}{n^2}\mathbb{E}\langle R_{1,0}^- (x_n x_{0n})^3\rangle + \frac{1}{n^3}\mathbb{E}\langle (x_{0n} x_n)^4\rangle.$$

$$\mathbb{E}\langle x_n x_{0n} (R_{1,0}^-)^3\rangle_t = \varphi(t).$$

(Cavity)

$$\Rightarrow \mathbb{E}\langle R_{1,0}^4\rangle = \lambda\,\mathbb{E}\langle R_{1,0}^4\rangle + \frac{K}{n^2}$$

$$+ K\,\mathbb{E}\langle R_{1,0}^5\rangle + \frac{K}{n}\mathbb{E}\langle R_{1,0}^3\rangle + o\left(\frac{1}{n^2}\right)$$

$$\mathbb{E}\langle R_{1,0}^5\rangle \leq \varepsilon\,\mathbb{E}\langle R_{1,0}^4\rangle + Ke^{-cn}.$$

Put everything together: $\forall \lambda < \lambda_c$

$$\mathbb{E}\langle R_{1,0}^4\rangle \leq \frac{K}{n^2}.$$

Main feature: recursive nature of the cavity method control $\mathbb{E}\langle R_{1,0}^{k}\rangle$ and you know that

$R_{1,0} \le \varepsilon$ w.h.p.

$\Rightarrow$ control $\mathbb{E}\langle R_{1,0}^{k+2}\rangle$.

- Prove convergence in probability:

$$\mathbb{E}\langle \mathbb{1}\{|R_{1,0}| = q\}\rangle$$

$$= \mathbb{E}\left[\frac{\int \mathbb{1}\{|R_{1,0}|=q\}\, e^{H(x)}\, dP_0^n(x)}{\int e^{H(x)}\, dP_0^n(x)}\right]$$

$$= \mathbb{E}\left[\frac{\exp\left(n\cdot \frac{1}{n}\log \int \text{———}\right)}{\exp\left(n\cdot \frac{1}{n}\log \int \text{———}\right)}\right]$$

$$= \mathbb{E}\left[\exp\left(n\cdot\left(f_n(q) - f_n\right)\right)\right].$$

where $f_n(q) = \frac{1}{n}\log \int \mathbb{1}\{R_{1,0} = q\}\, e^{H(x)}\, dP_0^n(x).$

$f_n = \frac{1}{n}\log\int e^{H(x)}\, dP_0^n(x)$

$$\le \exp\left(n\cdot\left(\mathbb{E} f_n(q) - \mathbb{E} f_n\right)\right) + e^{-cn}.$$
$$+\delta$$

If q is such that $\mathbb{E} f_n(q) < \mathbb{E} f_n - \varepsilon$
$\varepsilon > 0$.

$\Rightarrow \mathbb{E}\langle \mathbb{1}\{|R_{1,0}| = q\}\rangle \le e^{-cn}$.

suffices to show that $\forall\, q \neq 0$.

$$\mathbb{E} f_n(q) \leq \mathbb{E} f_n - c \quad , \quad c > 0 .$$