STSCI 6940:

Lecture 15 Date: 04/01

- Theorem: for $\lambda < \lambda_c$, P_0: centered, unit var, bounded

$$\log L \xrightarrow{d} \mathcal{N}(\pm\mu, \sigma^2)$$

$$\mu = \frac{1}{4}\left(-\log(1-\lambda) - \lambda\right).$$

$$\sigma^2 = \frac{1}{2}\left(-\log(1-\lambda) - \lambda\right). \quad \sigma^2 = 2\mu$$

"+" is under $\mathbb{P}_\lambda$, "−" is under $\mathbb{P}_0$.

- The characteristic function of $\log L$:

$$\phi_n(s) = \mathbb{E}_{\mathbb{P}_\lambda}\left[e^{is\log L}\right].$$

Proposition: $\forall \lambda < \lambda_c$, $\forall s \in \mathbb{R}$, $\exists K > 0$,

$$\left|\phi_n(s) - e^{(is - s^2)\mu}\right| \le \frac{K}{\sqrt{n}}.$$

$s \to e^{(is-s^2)\mu}$ is the characteristic fct of $N(\mu, 2\mu)$.

- Let's start looking at the KL divergence.

$$d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) \longrightarrow \mu = \frac{1}{4}\left(-\log(1-\lambda) - \lambda\right).$$

$$= \mathbb{E}_{\mathbb{P}_\lambda} \log Z(Y;\lambda)$$

$$Z(Y;\lambda) = \int e^{\sum_{i<j} \sqrt{\frac{\lambda}{n}} Y_{ij} x_i x_j - \frac{\lambda}{2n} x_i^2 x_j^2} \, dP_0^n(x).$$

$$y_{ij} = \sqrt{\frac{\lambda}{n}}\, x_{0i} x_{0j} + w_{ij}$$

$$\frac{d}{d\lambda} d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) = \sum_{i<j} \frac{1}{2\sqrt{\lambda n}} \mathbb{E} W_{ij} \langle x_i x_j \rangle + \frac{1}{n} \mathbb{E}\langle x_i x_j x_{0i} x_{0j} \rangle - \frac{1}{2n} \mathbb{E}\langle x_i^2 x_j^2 \rangle .$$

$$= \sum_{i<j} \frac{1}{2n} \mathbb{E}\left[\langle x_i^2 x_j^2 \rangle - \langle x_i x_j \rangle^2 \right].$$

$$+ \frac{1}{n} \mathbb{E}\langle x_i x_j x_{0i} x_{0j} \rangle$$

$$- \frac{1}{2n} \mathbb{E}\langle x_i^2 x_j^2 \rangle$$

$$\mathbb{E}\langle x_{0i} x_{0j} x_i x_j \rangle = \mathbb{E}\langle x_i x_j \rangle^2$$

$$= \mathbb{E}\Big[\mathbb{E}\big[x_{0i} x_{0j} \mathbb{E}[x_i x_j \mid Y] \mid Y\big]\Big] = \mathbb{E}\left[\mathbb{E}[x_i x_j \mid Y]^2 \right]$$

$$= + \frac{1}{2n} \sum_{i<j} \mathbb{E}\left[\langle x_{i0} x_{0j} x_i x_j \rangle\right]$$

$$= \frac{1}{4} \mathbb{E}\left[\left(n\langle R_{10}^2 \rangle - \langle x_1^2 x_{01}^2 \rangle \right) \right].$$

$$R_{10} = \frac{1}{n} \sum_{i=1}^{n} x_i x_{0i}$$

$$\begin{cases} n\,\mathbb{E}\langle R_{1,0}^2 \rangle \longrightarrow \dfrac{1}{1-\lambda}. & \forall \lambda < \lambda_c. \quad (*) \\ \mathbb{E}\langle x_1^2 x_{01}^2 \rangle \longrightarrow 1 \end{cases}$$

Assume $(*)$ $\quad \frac{d}{d\lambda} d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) \longrightarrow \frac{1}{4}\left(\frac{1}{1-\lambda} - 1 \right)$

Integrate w.r.t. λ: $d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) \longrightarrow \frac{1}{4}\left(-\log(1-\lambda) - \lambda\right)$

$$= \mu .$$

- Same strategy for characteristic function.

$$\phi_n(\lambda) = \mathbb{E}_{\mathbb{P}_\lambda} e^{is \log L(Y;\lambda)} \qquad s \in \mathbb{R} \text{ fixed}$$

Take a derivative w.r.t λ:

$$\phi_n'(\lambda) = is\, \mathbb{E}\Big[\Big\langle \Big(\sum_{i<j} \frac{1}{2\sqrt{\lambda n}} w_{ij} x_i x_j + \frac{1}{n} x_i x_j x_{0i} x_{0j} - \frac{1}{2n} x_i^2 x_j^2\Big) \Big\rangle \cdot e^{is\log L}\Big]$$

$$= \frac{1}{2}(is - s^2)\, \mathbb{E}\Big[\big(n\langle R_{10}^2\rangle - \langle x_1^2 x_{01}^2 \rangle\big) e^{is\log L}\Big]$$

$$\approx \mathbb{E}\Big[\qquad \Big] \cdot \mathbb{E}\big[e^{is\log L}\big]$$

$$\approx \frac{1}{2}(is - s^2)\, \frac{\lambda}{1-\lambda} \cdot \phi_n(\lambda).$$

$\phi_n(0) = 1 \Rightarrow$ Integrate diff. eq:

$$\phi_n(\lambda) \approx e^{(is - s^2)\mu}$$

<u>Asymptotic decoupling</u>:

Proposition: $\forall\, \lambda < \lambda_c$:

$$\begin{cases} \mathbb{E}\big[n\langle R_{10}^2\rangle e^{is\log L}\big] = \dfrac{1}{1-\lambda}\, \mathbb{E}\big[e^{is\log L}\big] + \delta \\ \mathbb{E}\big[\langle x_1^2 x_{01}^2\rangle e^{is\log L}\big] = \mathbb{E}\big[e^{is\log L}\big] + \delta \end{cases}$$

where $|\delta| \leq \dfrac{K(\lambda)}{\sqrt{n}}$.

$$\Rightarrow \quad \phi_n'(\lambda) = \frac{is - s^2}{4} \cdot \underbrace{\left(\frac{1}{1-\lambda} - 1\right)}_{\frac{\lambda}{1-\lambda}} \phi_n(\lambda) + \delta'$$

$$|\delta'| \leq \frac{K(\lambda)}{\sqrt{n}} \quad , \quad \lambda \to K(\lambda) \nearrow \text{ on } [0, \lambda_c].$$

- Proposition 2 : $\forall \lambda < 1$: [self consistency].

$$\begin{cases} \mathbb{E}\left[n \langle R_{10}^2 \rangle e^{is \log L}\right] = \frac{1}{1-\lambda} \mathbb{E}\left[\langle x_1^2 x_{01}^2 \rangle e^{is \log L}\right] + \delta \\ \mathbb{E}\left[\langle x_1^2 x_{01}^2 \rangle e^{is \log L}\right] = \mathbb{E}\left[e^{is \log L}\right] + \delta \end{cases}$$

where $|\delta| \leq K(\lambda) . n . \mathbb{E} \langle |R_{10}^3| \rangle$.

- Proposition 3 : $\forall \lambda < \lambda_c$.

$$\mathbb{E} \langle |R_{10}^3| \rangle \leq \frac{K(\lambda)}{n^{3/2}}$$

Proof of proposition 2 using the cavity method.

$$\mathbb{E}\left[\left(n \langle R_{10}^2 \rangle - \langle x_n^2 x_{0n}^2 \rangle\right) e^{is \log L}\right]$$

$$= \mathbb{E}\left[\left(n \left\langle \frac{1}{n^2} \sum_{i,j} x_i x_j x_{0i} x_{0j} \right\rangle - \langle x_n^2 x_{0n}^2 \rangle\right) e^{is \log L}\right]$$

$$= \mathbb{E}\left[\left(\left(\frac{1}{n} \sum_{j=1}^{n} \left\langle x_j x_{0j} \left(\sum_{i=1}^{n} x_i x_{0i}\right)\right\rangle - \langle x_n^2 x_{0n}^2 \rangle\right)\right.\right.$$

$$= \mathbb{E}\left[\left(\left\langle x_n x_{0n} \left(\sum_{i=1}^{n} x_i x_{0i}\right)\right\rangle - \langle x_n^2 x_{0n}^2 \rangle\right) e^{is \log L}\right]$$

$$= \mathbb{E}\left[n < x_n x_{on} \left(\frac{1}{n}\sum_{i=1}^{n-1} x_i x_{oi}\right) > e^{is \log L} \right].$$

$$R_{10}^{-} = \frac{1}{n}\sum_{i=1}^{n-1} x_i x_{oi}$$

$$= \mathbb{E}\left[n < x_n x_{on} \cdot R_{10}^{-} > \cdot e^{is \log L} \right]$$

Interpolation :

$$H_t(x) = \sum_{1 \le i < j \le n-1} \sqrt{\frac{\lambda}{n}}\, w_{ij} x_i x_j + \frac{\lambda}{n} x_i x_j x_{oi} x_{oj} - \frac{\lambda}{2n} x_i^2 x_j^2$$

$$+ \sum_{i=1}^{n-1} \sqrt{\frac{\lambda t}{n}}\, w_{in} x_i x_n + \frac{\lambda t}{n} x_i x_n x_{oi} x_{on} - \frac{\lambda t}{2n} x_i^2 x_n^2 .$$

$0 \le t \le 1$

$$\begin{cases} H_1(x) = H(x). \\ H_0(x) = fct(x_1, \ldots, x_{n-1}). \end{cases}$$

Let $X(t) = \exp\left(is \log \int e^{H_t(x)} dP_0^n(x) \right).$

$$< \cdot >_t = \frac{\int \cdot\, e^{H_t(x)} dP_0^n(x)}{\int e^{H_t(x)} dP_0^n(x)} .$$

Consider the function,

$$\varphi(t) = n\, \mathbb{E}\left[< x_n x_{on} R_{10}^{-} >_t X(t) \right]$$

$$\varphi(1) = \mathbb{E}\left[n < x_n x_{on} \cdot R_{10}^{-} > \cdot e^{is \log L} \right].$$

$$\varphi(0) = n\, \mathbb{E}\left[\langle x_n x_{0n} R^-_{1,0} \rangle_0 \cdot X(0)\right].$$

$$= n\, \underset{x, x_0 \sim P_0 \text{ indep.}}{\mathbb{E}[x x_0]} \cdot \mathbb{E}\left[\langle R^-_{1,0}\rangle_0 \; X(0)\right]$$

$$= 0 .$$

$$\varphi'(t) = n \sum_{a,b} c_{a,b} \cdot \mathbb{E}\left[\langle x_n x_{0n} x_n^{(a)} x_n^{(b)} R^-_{1,0} R^-_{a,b} \rangle_t X(t)\right]$$

$(a,b) \in \{(1,0), (2,0), (1,2), (2,3)\}$.

$c_{a,b}$ coefficients where $c_{1,0} = \lambda$

$$\varphi'(0) = n\lambda \underset{x, x_0 \sim P_0 \text{ indep}}{\mathbb{E}[x^2 x_0^2]} \; \mathbb{E}\left[\langle (R^-_{1,0})^2 \rangle_0 \cdot X(0)\right]$$

$$= \lambda\, n\, \mathbb{E}\left[\langle (R^-_{1,0})^2 \rangle_0 \, X(0)\right].$$

$$\varphi''(t) = \lambda^2 n \sum_{a,b,c,d} c_{a,b,c,d}\, \mathbb{E}\Big[\langle x_n x_{0n} x_n^a x_{0n}^b x_n^c x_{0n}^d$$
$$R^-_{1,0} \cdot R^-_{a,b} \, R^-_{c,d} \rangle_t X(t)\Big]$$

$$\Rightarrow \quad |\varphi''(t)| \le K \cdot n\, \mathbb{E}\left[\langle |R^-_{1,0}|^3 \rangle_t\right].$$

$$\le K n\, \mathbb{E}\left[\langle |R^-_{1,0}|^3 \rangle_1\right].$$

(Gronwall's).

assuming prop 2:

$$\le \frac{K(\lambda)}{\sqrt{n}}.$$

$$\Rightarrow \quad |\underbrace{\varphi(1) - \varphi(0)}_{=0} - \varphi'(0)| \leq \sup_t |\varphi''(t)|$$

$$\leq \frac{K}{\sqrt{n}}.$$

$$\varphi(1) = n\,\mathbb{E}\left[\langle x_n x_{0n} R_{1,0}^- \rangle e^{is\log L}\right]$$

$$= \mathbb{E}\left[\left(n\langle (R_{1,0})^2\rangle - \langle x_n^2 x_{0n}^2\rangle\right) e^{is\log L}\right]$$

$$\varphi'(0) = \lambda n\,\mathbb{E}\left[\langle (R_{1,0}^-)^2\rangle_0\, X(0)\right].$$

relate $\varphi'(0)$ to $\lambda n\,\mathbb{E}\left[\langle (R_{1,0})^2\rangle_1\, X(1)\right]$

$$\psi(t) = \lambda n\,\mathbb{E}\left[\langle (R_{1,0}^-)^2\rangle_t\, X(t)\right].$$

$$|\psi'(t)| \leq \frac{K}{\sqrt{n}}.$$

$$\Rightarrow \quad |\psi(1) - \psi(0)| \leq \frac{K}{\sqrt{n}}.$$

$$\Rightarrow \quad \left|\varphi'(0) - \lambda n\,\mathbb{E}\left[\langle (R_{1,0})^2\rangle e^{is\log L}\right]\right| \leq \frac{K}{\sqrt{n}}.$$

$$\underset{(\varphi(1))}{\hookrightarrow} \mathbb{E}\left[\left(n\langle R_{1,0}^2\rangle - \langle x_n^2 x_{0n}^2\rangle\right) e^{is\log L}\right]$$

$$= \lambda n\,\mathbb{E}\left[\langle R_{1,0}^2\rangle e^{is\log L}\right] + \delta$$

$$|\delta| \leq \frac{K}{\sqrt{n}}.$$

$\Rightarrow$ If $\lambda < 1$: $n\,\mathbb{E}\left[\langle (R_{1,0})^2\rangle e^{is\log L}\right] =$

$$= \frac{1}{1-\lambda}\cdot \mathbb{E}\left[\langle x_n^2 x_{0n}^2\rangle e^{is\log L}\right] + \delta.$$