STSCI 6940:

Lecture 14 Date: 03/30

The second moment method:

$$Y_{ij} = \sqrt{\frac{\lambda}{n}} x_i x_j + w_{ij}, \quad i<j$$

$$L = \frac{dP_\lambda}{dP_0} = \int e^{\sum_{i<j} \sqrt{\frac{\lambda}{n}} Y_{ij} x_i x_j - \frac{\lambda}{2n} x_i^2 x_j^2} dP_0^n(x).$$

$$\mathbb{E}_0 [L^2] = \mathbb{E}_\lambda [L]$$

$$= \int \mathbb{E}_\lambda \left[e^{\sum_{i<j} \sqrt{\frac{\lambda}{n}} Y_{ij} x_i x_j - \frac{\lambda}{2n} x_i^2 x_j^2} \right] dP_0^n(x).$$

$(Y_{ij} = \sqrt{\frac{\lambda}{n}} x_{0i} x_{0j} + w_{ij})$

$$= \int \mathbb{E}\, e^{\sum_{i<j} \sqrt{\frac{\lambda}{n}} w_{ij} x_i x_j + \frac{\lambda}{n} x_i x_j x_{0i} x_{0j} - \frac{\lambda}{2n} x_i^2 x_j^2} dP_0^n(x).$$

$$= \iint e^{\sum_{i<j} \frac{\lambda}{n} x_i x_j x_i' x_j'} dP_0^n(x)\, dP_0^n(x')$$

$$= \mathbb{E}_{x,x' \sim P_0^n} \left[e^{\frac{1}{2} \cdot \frac{\lambda}{n} \langle x, x' \rangle^2 - \frac{1}{2} \sum_{i=1}^n \frac{\lambda}{n} x_i^2 x_i'^2} \right].$$

$$\leq e^{-}$$

Include diagonal terms:

$$Y_{ii} = \sqrt{\frac{\lambda}{n}} x_i^2 + W_{ii}, \quad W_{ii} \sim N(0,2).$$

$$\Rightarrow \mathbb{E}_0 [L^2] = \mathbb{E}_{x,x' \sim P_0^n} \left[e^{\frac{\lambda}{2n} \langle x, x' \rangle^2} \right].$$

$$= \mathbb{E}_{x,x'} \left[e^{\frac{\lambda}{2} \left(\frac{\langle x, x' \rangle}{\sqrt{n}} \right)^2} \right].$$

$\mathbb{E} \left(\frac{\langle x, x' \rangle}{\sqrt{n}} \right)^2 = \sigma^2$

$$\to \mathbb{E}_g \left[e^{\frac{\lambda}{2} g^2 \sigma^4} \right]$$

$$= \frac{1}{|1 - \lambda\sigma^2|^{\frac{1}{2}}}.$$

- $\langle x, x' \rangle$ to be of order $\sqrt{n}$ w.h.p.

- Assume $\langle x, x' \rangle$ is $\sigma\sqrt{n}$ - subgaussian.

$$\left(\mathbb{E}\left[e^{\theta \langle x, x' \rangle} \right] \leq e^{\frac{\theta^2 \sigma^2 n}{2}} \quad \forall \theta \in \mathbb{R}. \right)$$

Then $\mathbb{E}\left[e^{\frac{\lambda}{2n} \langle x, x' \rangle^2} \right]$

$$= \mathbb{E}\left[e^{\sqrt{\frac{\lambda}{n}} \langle x, x' \rangle \cdot g} \right], \quad g \sim N(0,1).$$

$$\leq \mathbb{E}\, e^{\frac{\lambda}{2n} g^2 \sigma^2 n}.$$

$$= \mathbb{E}\left[e^{\frac{\lambda}{2} g^2 \sigma^2} \right]$$

$$= \frac{1}{\sqrt{1 - \lambda\sigma^2}}.$$

$$\mathbb{E}\left[e^{\theta g} \right] = e^{\theta^2/2}.$$

If $\lambda\sigma^2 < 1 \Rightarrow \mathbb{E}_0\left[L^2 \right] < +\infty$,

$\Rightarrow \mathbb{P}_\lambda \triangleleft \mathbb{P}_0$.

Simplify assumption to: $\mathbb{P}_0$ is σ^2 - subgaussian.
If $\lambda\sigma^2 < 1 \Rightarrow \mathbb{P}_\lambda \triangleleft \mathbb{P}_0$.

Proof relies on conditional second moment.

The event on which we condition:

Fix $\delta > 0$

$$(x \sim P_0^n) \qquad \mathcal{E} = \left\{ \|x\|_2 \le \sqrt{n(1+\delta)} \right\}$$

$$\mathbb{P}_\lambda(\mathcal{E}) \ge 1 - \varepsilon .$$

$$\mathbb{E}\left\{ \|x\|_2^2 \right\} = n .$$

$$\mathbb{P}\Big(\underbrace{\|x\|_2^2}_{\sum x_i^2} - n \ge \delta n \Big) \le \frac{\mathbb{E}\left[\left(\sum x_i^2 - n \right)^2 \right]}{\delta^2 n^2}$$

$$= \frac{\mathbb{E}\left[(x_1^2 - 1)^2 \right]}{\delta^2 n} \xrightarrow[n \to +\infty]{} 0 .$$

$$\Rightarrow \quad \mathbb{P}(\mathcal{E}) \ge 1 - \frac{a}{\delta^2 n} .$$

$$\tilde{\mathbb{P}}_\lambda(\cdot) = \mathbb{P}_\lambda(\cdot \mid \mathcal{E}) .$$

- $$\mathbb{E}_{P_0}\left\{ \tilde{L}_\lambda^2 \right\} = \mathbb{E}_{x,x' \sim (P_0^n)^2}\left\{ e^{\frac{\lambda}{2n}\langle x, x' \rangle^2} \;\middle|\; \begin{array}{l} \|x\|_2^2 \le (1+\delta)n \\ \|x'\|_2^2 \le (1+\delta)n \end{array} \right\}$$

$$\tilde{L}_\lambda = \frac{d\tilde{\mathbb{P}}_\lambda}{d\mathbb{P}_0}$$

$$\le \mathbb{E}_{x,x'}\left[e^{\frac{\lambda}{2n}\langle x, x' \rangle^2} \;\middle|\; \|x\|_2^2 \le (1+\delta)n \right]$$

$$= \mathbb{E}_{x,x',g}\left\{ e^{\sqrt{\frac{\lambda}{n}}\langle x, x' \rangle g} \;\middle|\; \|x\|_2^2 \le (1+\delta)n \right\}$$

$$\leq \mathbb{E}_{x,g}\left[e^{\frac{\lambda}{2n}\|x\|^2 g^2 \sigma^2} \;\middle|\; \|x\|_2^2 \leq (1+\delta)n \right]$$

$$\leq \mathbb{E}_{g}\left[e^{\frac{\lambda}{2n}(1+\delta)n \cdot g^2\sigma^2} \right]$$

$$\leq \frac{1}{\sqrt{1-(1+\delta)\lambda\sigma^2}} \cdot < +\infty$$

$\Rightarrow$ If $(1+\delta)\lambda\sigma^2 < 1$ then $\tilde{\mathbb{P}}_\lambda \lhd \mathbb{P}_0$.

$$\Rightarrow \mathbb{P}_1(A_n) \leq \sqrt{\mathbb{E}_0[\tilde{L}_n^2] \cdot \mathbb{P}_0(A_n)} + \mathbb{P}_1(\mathcal{E}^c)$$

$$\leq \sqrt{K \cdot \mathbb{P}_0(A_n)} + \frac{a}{\delta n}.$$

$$\longrightarrow 0 \text{ if } \mathbb{P}_0(A_n) \to 0.$$

If $\lambda\sigma^2 < 1$ then $\mathbb{P}_\lambda \lhd \mathbb{P}_0$.

$$\mathbb{E}\left[f(\langle x, x' \rangle^2) \;\middle|\; \|x\|_2^2 \leq n(1+\delta) \right].$$

$$\mathbb{E}\left[\rule{4cm}{0.4pt} \;\middle|\; \rule{2cm}{0.4pt}, \|x'\|_2^2 \leq n(1+\delta) \right]$$

$$\cdot \underbrace{\mathbb{P}\left(\|x'\|_2^2 \leq n(1+\delta) \right)}_{\geq 1 - \frac{a}{\delta n}}.$$

$$+ \mathbb{E}\left[\quad \middle|\; \rule{2cm}{0.4pt}, \|x'\|_2^2 > n(1+\delta) \right]$$

$$\cdot \underbrace{\mathbb{P}\left(\|x'\|_2^2 \geq n(1+\delta) \right)}_{\frac{a}{\delta^2 n}}.$$

$$\tilde{\mathbb{P}}_\lambda (Y \mid \mathcal{E}) \qquad \mathcal{E} = \{ \|x\|_2^2 \le n(1+\varepsilon) \}.$$

$$Y = \sqrt{\frac{\lambda}{n}}\, x x^T + W$$

Second moment method implies contiguity for all $\lambda < \frac{1}{\sigma^2}$, σ^2: subgaussian constant of the prior.

- For Rademacher: $\sigma^2 = 1$

$$\mathbb{P}_\lambda \triangleleft \mathbb{P}_0 \text{ for all } \lambda < 1.$$

$$\lambda > 1 : \text{ detection is possible.}$$

Second is tight for sharp subgaussian priors:

sharp subgaussian : $\sigma^2 = \text{variance} = 1$.

$\sigma^2 \geq$ variance.

- Sparse Rademacher : $P_0 = \frac{\rho}{2}\delta_{+\frac{1}{\sqrt{\rho}}} + (1-\rho)\delta_0 + \frac{\rho}{2}\delta_{-\frac{1}{\sqrt{\rho}}}$.

Lemma (PBWM 2016):

If $\rho \geq \frac{1}{\sqrt{3}}$ then $\sigma^2 = 1$

If $\rho < \frac{1}{\sqrt{3}}$ then $\sigma^2 > 1$.

$$\sigma^2 = \sup_{t \in \mathbb{R}} \left\{ \frac{2}{t^2} \log\left(1 - \rho^2 + \frac{\rho^2}{2} e^{t/\rho} + \frac{\rho^2}{2} e^{-t/\rho}\right) \right\}$$

$\Rightarrow$ contiguity up to $1/\sigma^2$.

By simulation one sees $\frac{1}{\sigma^2} < \lambda_c$ for $\rho < \frac{1}{\sqrt{3}}$.
sharper results?

- Fluctuations of likelihood ratio.

Le Cam's first lemma:

1. If $\frac{d\mathbb{P}_n}{d\mathbb{Q}_n} \xrightarrow[\text{under } \mathbb{Q}_n]{d} e^{N(\mu,\sigma^2)}$, $\mu = -\frac{1}{2}\sigma^2$.

 then $\mathbb{P}_n \triangleleft\triangleright \mathbb{Q}_n$.

2. If $\frac{d\mathbb{P}_n}{d\mathbb{Q}_n} \xrightarrow[\text{under } \mathbb{P}_n]{d} e^{N(\mu,\sigma^2)}$ $\mu = \frac{1}{2}\sigma^2$

 then $\mathbb{P}_n \triangleleft\triangleright \mathbb{Q}_n$.

$$U = e^Z, \quad Z \sim N(\mu, \sigma^2).$$

$$\mathbb{E}\, U = e^{\mu + \frac{\sigma^2}{2}} = 1 \quad \text{if } \mu = -\frac{\sigma^2}{2}.$$

- Theorem: For $\lambda < \lambda_c$, $\mathbb{P}_0$: centered, unit var, bounded.

$$\log L \xrightarrow{d} N(\pm\mu, \sigma^2)$$

$$\mu = \frac{1}{4}\left(-\log(1-\lambda) - \lambda\right).$$

$$\sigma^2 = \frac{1}{2}\left(-\log(1-\lambda) - \lambda\right).$$

"+" is under $\mathbb{P}_\lambda$, "−" is under $\mathbb{P}_0$.

$\Rightarrow$ Implies contiguity up to λ_c.

- The characteristic function of $\log L$:

$$\phi_n(s) = \mathbb{E}_{\mathbb{P}_\lambda}\left[e^{is \log L}\right].$$

Proposition: $\forall \lambda < \lambda_c$, $\forall s \in \mathbb{R}$, $\exists K > 0$,

$$\left| \phi_n(s) - e^{(is - s^2)\mu} \right| \leq \frac{K}{\sqrt{n}},$$

$s \to e^{(is - s^2)\mu}$ is the charateristic fct of $\mathcal{N}(\mu, \sigma^2)$

$$\mu = \frac{1}{2}\sigma^2.$$

- Corollaries of proof : If $\lambda < \lambda_c$

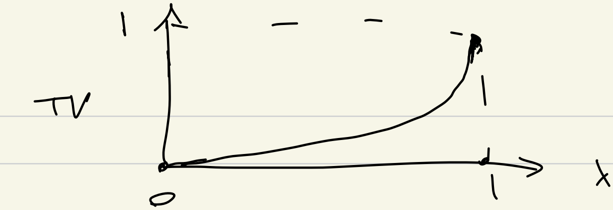
$$\begin{cases} d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) \xrightarrow[n \to +\infty]{} \mu. \\ d_{TV}(\mathbb{P}_\lambda, \mathbb{P}_0) \longrightarrow 1 - \operatorname{erfc}(\sqrt{\mu}). \end{cases}$$

$$\mu = \frac{1}{4}\left(-\log(1-\lambda) - \lambda\right).$$

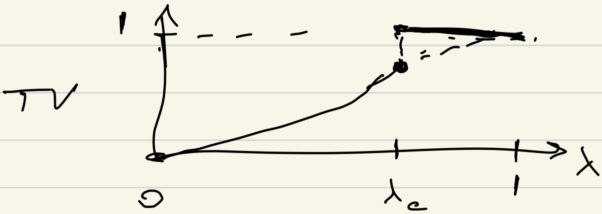
$$\frac{1}{n} d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) \to \phi_{RS}(\lambda)$$

$$\begin{cases} \text{If } \lambda > \lambda_c : & d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) = \Theta(n). \\ \text{If } \lambda < \lambda_c : & d_{KL}(\mathbb{P}_\lambda, \mathbb{P}_0) = o(n) \end{cases}$$

$$\operatorname{efrc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2}\, dt.$$



if $\lambda_c = 1$



if $\lambda_c < 1$