STSCI 6940:

Lecture 11 — Date: 03/18

$$E_t = \{ A q_\perp^s = y^s \quad 0 \leq s \leq t-1, \} .$$

$$A|_{\mathfrak{F}_t} = A|_{E_t} \quad (\text{because } A \sim GOE(n))$$

$$\stackrel{d}{=} \mathbb{E}[A \mid \mathfrak{F}_t] + \mathcal{P}_t(A^{new})$$

$$A^{new} \stackrel{d}{=} A, \text{ is indep from } A.$$

$\mathcal{P}_t$: orthogonal proj onto $E_t^\perp$

$$= \{ A \in \mathbb{R}^{n \times n} : A = A^T, \ A q_\perp^s = 0 \ \forall \ s \leq t-1 \}$$

Claim: • $\mathbb{E}[A \mid \mathfrak{F}_t] = A - \tilde{P}_{t-1}^\perp A \tilde{P}_{t-1}^\perp$

$(\tilde{P}_{t-1} = P_{t-1})$ $\qquad = A - P_{t-1}^\perp A P_{t-1}^\perp .$

• $\mathcal{P}_t(A) = \tilde{P}_{t-1}^\perp A \tilde{P}_{t-1}^\perp = P_{t-1}^\perp A P_{t-1}^\perp .$

$$\Rightarrow h^{t+1}|_{\mathfrak{F}_t} \stackrel{d}{=} P_{t-1}^\perp A|_{\mathfrak{F}_t} P_{t-1}^\perp q_t + H_{t-1} \alpha^t$$

$$\stackrel{d}{=} P_{t-1}^\perp \left(\mathbb{E}[A \mid \mathcal{G}_{t-1}] + \mathcal{P}_t(A^{new}) \right) P_{t-1}^\perp q_t + H_{t-1} \alpha^t .$$

$$= P_{t-1}^\perp A^{new} P_{t-1}^\perp q_t + H_{t-1} \alpha^t .$$

Proposition 2 : Geometry of iterates of LAMP:

1. $\forall\, t \geq 0$, $0 \leq r, s \leq t$:

$$\frac{1}{n} < h^{r+1}, h^{s+1} > \overset{P}{\simeq} \frac{1}{n} < q^r, q^s > .$$

$a_n \overset{P}{\simeq} b_n$ means $a_n - b_n \longrightarrow 0$ in prob.

2. $$\frac{1}{n} \sum_{i=1}^{n} \phi (h_i^0, \ldots, h_i^t) \xrightarrow[n \to +\infty]{P} \mathbb{E}\phi (Z^0, \ldots, Z^t)$$

$(Z^0, \ldots, Z^t)$ has the same law as in

SE thm : $\mathbb{E}[Z^{t+1} Z^{s+1}] = \mathbb{E}[f_t(Z^t) f_s(Z^s)]$

Proof : By induction. (prove 1 then 2)

1. $t = 1$

$$\frac{1}{n} < h^1, h^1 > = \frac{1}{n} \| h^1 \|_2^2 = \frac{1}{n} \| A q^0 \|_2^2$$

$$\simeq \frac{1}{n} \| q^0 \|_2^2 \qquad \blacksquare$$

Assume statement up to time t.

$\forall\, 0 \leq r, s \leq t$: $\frac{1}{n} < h^{r+1}, h^{s+1} > \simeq \frac{1}{n} < q^r, q^s >$.

- $s \leq t-1$: $\frac{1}{n} < h^{t+1}, h^{s+1} > / \mathfrak{F}_t$

$$\overset{d}{=} \frac{1}{n} < P_{t-1}^{\perp} A^{new} P_{t-1}^{\perp} q_t + H_{t-1} \alpha^t, h^{s+1} > .$$

$$= \frac{1}{n} \langle \overset{I - P_{t-1}}{P_{t-1}^{\perp}} A^{new} q_{\perp}^{t}, h^{s+1} \rangle + \frac{1}{n} \sum_{r=0}^{t} \alpha_{r-1}^{t} \langle h^{r}, h^{s+1} \rangle$$

rank of $(P_{t-1}^{\perp}) \geq n - t$

P_{t-1} projects on a space of dimension $\leq t$

$q_0, \ldots, q^{t-1}$

$$\approx 0 + \frac{1}{n} \sum_{r=0}^{t} \alpha_{r-1}^{t} \langle q^{r-1}, q^{s} \rangle .$$

$$= \frac{1}{n} \langle \sum_{r=0}^{t} \alpha_{r-1}^{t} q^{r-1}, q^{s} \rangle .$$

$$= \frac{1}{n} \langle q_{\parallel}^{t}, q^{s} \rangle .$$

$\langle q_{\perp}^{t}, q^{s} \rangle = 0$

$$= \frac{1}{n} \langle q^{t}, q^{s} \rangle$$

- $s = t$:

$$\frac{1}{n} \| h^{t+1} \|_2^2 \overset{d}{=} \frac{1}{n} \| P_{t-1}^{\perp} A^{new} q_{\perp}^{t} + H_{t-1} \alpha^{t} \|_2^2$$

$$= \frac{1}{n} \| P_{t-1}^{\perp} A^{new} q_{\perp}^{t} \|_2^2 + \frac{1}{n} \| \sum_{s=0}^{t} \alpha_{s-1}^{t} h^{s} \|_2^2$$

$$\approx \frac{1}{n} \| q_{\perp}^{t} \|_2^2 + \sum_{s,r=0}^{t} \alpha_{s-1}^{t} \alpha_{r-1}^{t} \langle h^{r}, h^{s} \rangle / n$$

$$\simeq \frac{1}{n} \| q_{\perp}^{t} \|_2^2 + \sum_{s,r=0}^{t} \alpha_{s-1}^{t} \alpha_{r-1}^{t} \langle q^{r-1}, q^{s-1} \rangle / n$$

$$= \frac{1}{n} \| q_{\perp}^{t} \|_2^2 + \frac{1}{n} \| \underbrace{\sum_{s=0}^{t} \alpha_{s-1}^{t} q^{s-1}}_{q_{\parallel}^{t}} \|_2^2$$

$$= \frac{1}{n} \| q^{t} \|_2^2 .$$

Cross term: $\frac{1}{n} \langle P_{t-1}^{\perp} A^{new} q_{\perp}^{t}, H_{t-1} \alpha^{t} \rangle$.

$$\frac{1}{n} \sum_{i=1}^{n} g_i x_i = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

- Proof of SE for LAMP.

Assume $\frac{1}{n} \sum_{i=1}^{n} \phi(h_i^0, \ldots, h_i^t) = \mathbb{E}\,\phi(Z^0, \ldots, Z^t)$.

$$\langle \phi(h^0, \ldots, h^t) \rangle_n$$

$$\langle \phi(h^0, \ldots, h^t, h^{t+1}) \rangle_n$$

$$\overset{d}{=} \langle \phi(h^0, \ldots, h^t, P_{t-1}^{\perp} A^{new} P_{t-1}^{\perp} q^t + H_{t-1} \alpha^t) \rangle_n$$

$$\overset{d}{\approx} \langle \phi(h^0, \ldots, h^t, \tilde{g} + \sum_{s=0}^{t} \alpha_{s-1}^{t} h^s) \rangle_n$$

where $\tilde{g} \sim N(0, \frac{1}{n} \| q_{\perp}^t \|_2^2 \cdot I)$,

(By induction)

$$\simeq \mathbb{E}\,\phi(Z^0, \ldots, Z^1, g + \sum_{s=0}^{t} \alpha_{s-1}^{t} Z^s)$$

$$g \sim N(0, \frac{1}{n} \| q_{\perp}^t \|_2^2) \quad \text{(univariate)}.$$

$$\tilde{Z}^{t+1} = g + \sum_{s=0}^{t} \alpha_{s-1}^{t} Z^s.$$

we want: $(Z^0, \ldots, Z^t, \tilde{Z}^{t+1}) \overset{d}{=} (Z^0, \ldots, Z^t, Z^{t+1})$.

It suffices to prove:

$$\mathbb{E}[\tilde{Z}^{t+1} \cdot Z^s] = \mathbb{E}[Z^{t+1} \cdot Z^s] \quad \forall\, s \le t.$$

and $\mathbb{E}[(\tilde{Z}^{t+1})^2] = \mathbb{E}[(Z^{t+1})^2]$.

$$\mathbb{E}[(\tilde{Z}^{t+1})^2] \simeq \frac{1}{n} \| \tilde{g} + \sum_{s=0}^{t} \alpha^t_{s-1} h^s \|_2^2$$

$$\simeq \frac{1}{n} \|\tilde{g}\|_2^2 + \frac{1}{n} \| \sum_{s=0}^{t} \alpha^t_{s-1} h^s \|_2^2$$

$$\simeq \frac{1}{n} \|q^t_\perp\|_2^2 + \frac{1}{n} \|q_{\|}\|_2^2$$

$q^t = f_t(h^t)$

$\frac{1}{n}\|q^t\|_2^2 \simeq \mathbb{E}[f_t^2]$

$= \mathbb{E}[(Z^{t+1})^2]$

$$= \frac{1}{n} \| q^t \|_2^2 .$$

induction

$$\simeq \mathbb{E}[(Z^{t+1})^2] .$$

- $\mathbb{E}[\tilde{Z}^{t+1} \cdot Z^r] \simeq \frac{1}{n} \langle \tilde{g} + \sum_{s=0}^{t} \alpha^t_{s-1} h^s, h^r \rangle$

$r \leq t$

$$= \sum_{s=0}^{t} \alpha^t_{s-1} \langle h^s, h^r \rangle / n$$

geometry of iterates $\Downarrow$

$$\simeq \sum_{s=0}^{t} \alpha^t_{s-1} \langle q^{s-1}, q^{r-1} \rangle / n .$$

$$= \frac{1}{n} \langle q^t_{\|}, q^{r-1} \rangle$$

$$= \frac{1}{n} \langle q^t, q^{r-1} \rangle .$$

induction $\Downarrow$

$$= \mathbb{E}[f_t(Z^t) \cdot f_{r-1}(Z^{r-1})] .$$

$$= \mathbb{E}[Z^{t+1} \cdot Z^r] .$$

$\blacksquare$.

$$(Z^0, \ldots, Z^t, \tilde{Z}^{t+1}) \overset{d}{=} (Z^0, \ldots, Z^{t+1}) .$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^{n} \phi(h_i^0, \ldots, h_i^{t+1}) \to \mathbb{E}\, \phi(Z^0, \ldots, Z^{t+1}) .$$

$$q^t = f_t(h^t). \qquad r \leq t$$

$$\frac{1}{n} \langle q^t, q^{r-1} \rangle = \frac{1}{n} \langle f_t(h^t), f_{r-1}(h^{r-1}) \rangle.$$

By induction we know the law (h^t, h^{r-1}).

$$\Rightarrow \qquad \simeq \mathbb{E}\left[f_t(Z^t) \, f_{r-1}(Z^{r-1}) \right].$$

, we want to go back to AMP.

Proposition 1:

LAMP and AMP trajectories are close:

$$\begin{cases} \frac{1}{\sqrt{n}} \| h^{t+1} - x^{t+1} \|_2 \xrightarrow{P} 0 & \forall t \\ \frac{1}{\sqrt{n}} \| q^t - m^t \|_2 \xrightarrow{P} 0 & \forall t. \end{cases}$$

we need to define an intermediate iteration:

$$\begin{cases} \hat{h}^{t+1} = A q^t - b_t q^{t-1}, \\ \hat{h}^1 = h^1. \end{cases}$$ wher

$\{q^t\}$ comes from LAMP

Lemma : $\frac{1}{\sqrt{n}} \| \hat{h}^{t+1} - h^{t+1} \|_2 \xrightarrow{P} 0.$

Let's assume Lemma :

Recall AMP iteration:

$$\begin{cases} x^{t+1} = A\, m^t - b_t\, m^{t-1} \\ m^t = f_t(x^t). \end{cases}$$

$$\begin{cases} \hat{h}^{t+1} - x^{t+1} = A(q^t - m^t) - b_t(q^{t-1} - m^{t-1}). \\ m^t - q^t = f_t(x^t) - f_t(h^t). \end{cases}$$

$$\Rightarrow \|h^{t+1} - x^{t+1}\|_2 \leq \|h^{t+1} - \hat{h}^{t+1}\|_2 + \|\hat{h}^{t+1} - x^{t+1}\|_2$$

$$\leq \|\hat{h}^{t+1} - h^{t+1}\|_2 + \|A\|_{op} \cdot \|q^t - m^t\|_2 + |b_t| \cdot \|q^{t-1} - m^{t-1}\|_2$$

$$\|q^t - m^t\|_2 = \|f_t(x^t) - f_t(h^t)\|_2$$

$$\leq L_t \cdot \|x^t - h^t\|_2.$$

By induction: $h^1 = \hat{h}^1 = x^1$

If assume that $\frac{1}{\sqrt{n}}\|m^s - q^s\|_2 \longrightarrow 0 \quad \forall s \leq t$

$$\frac{1}{\sqrt{n}}\|h^s - x^s\|_2 \longrightarrow 0 \quad \forall s \leq t$$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{n}}\|h^{t+1} - x^{t+1}\|_2 \longrightarrow 0 \\ \frac{1}{\sqrt{n}}\|m^{t+1} - q^{t+1}\|_2 \longrightarrow 0 \end{cases}$$

$$f_{t+1}(x^{t+1}) - f_{t+1}(h^{t+1}).$$

Prove

Lemma : $\frac{1}{\sqrt{n}} \| \hat{h}^{t+1} - h^{t+1} \|_2 \xrightarrow{P} 0$.

by induction. $\hat{h}^1 = h^1$

After (non trivial) manipulations, we get

$$\circledast \quad \hat{h}^{t+1} - h^{t+1} = Q_{t-1} (Q_{t-1}^T Q_{t-1})^{-1} (\hat{H}_{t-1} - H_{t-1})^T \cdot q_\perp^t + \sum_{s=0}^{t} \alpha_s^t (\hat{h}^s - h^s)$$

$$\hat{H}_{t-1} = [\hat{h}^1 | \ldots | \hat{h}^t]$$

Next lecture we prove $\circledast$.

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we say $(h^0, \ldots, h^t) \overset{d}{=} (Z^0, \ldots, Z^t)$

in the sense $\frac{1}{n} \sum_{i=1}^{n} \phi(h_i^0, \ldots, h_i^t) \longrightarrow \mathbb{E}\phi(Z^0, \ldots, Z^t)$