STSCI 6940:

Lecture 10 — Date: 03/16

Elements of rigorous proof of State evolution.

$A_{ij} = A_{ji} \sim N(0, \frac{1}{n})$, $A_{ii} \sim N(0, \frac{2}{n})$.

$A \sim GOE(n)$.

AMP:
$$\begin{cases} x^{t+1} = A m^t - b_t \cdot m^{t-1} \\ m^t = f_t(x^t), \quad b_t = \frac{1}{n}\sum_{i=1}^{n} f_t'(x_i^t) \end{cases}$$
$$= \langle f_t'(x^t) \rangle.$$

Init: $m^{-1} = 0$, $x^0 \in \mathbb{R}^n$.

- The empirical dist $\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i^0} \rightarrow P_0$ has mean, finite second moment.
- f_t is lipscitz $\forall t$.

State evolution: For any $\phi: \mathbb{R}^{t+1} \rightarrow \mathbb{R}$ pseudo-Lipschitz then
$$\frac{1}{n}\sum_{i=1}^{n} \phi(x_i^0, \dots, x_i^t) \xrightarrow[n \to +\infty]{P} \mathbb{E}\,\phi(Z^0, \dots, Z^t).$$

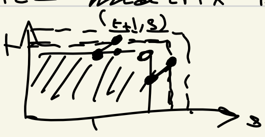
where $Z^0 \sim P_0$, $(Z^1, \dots, Z^t)$ is a Gaussian vector centered, covariance defined recursively:
$$\forall t, s \geq 0 \quad \mathbb{E}\left[Z^{t+1} \cdot Z^{s+1} \right] = \mathbb{E}\left[f_t(Z^t) \cdot f_s(Z^s) \right].$$

Z^0 and $(Z^1, \dots, Z^t)$ are independent.

E.g. $s = 0$: $\mathbb{E}[Z^{t+1} \cdot Z^1] = \mathbb{E}[f_t(Z^t) \cdot f_0(Z^0)]$ indep
$$= \mathbb{E}[f_t(Z^t)] \cdot \mathbb{E}[f_0(Z^0)]$$

The covariance matrix is well defined by induction

$(t, s-1)$



- If ϕ depends only on (x_i^t), then

$$\frac{1}{n} \sum_{i=1}^{n} \phi(x_i^t) \longrightarrow \mathbb{E}\, \phi(Z^t).$$

$$Z^t \sim N(0, \sigma_t^2).$$

$$\sigma_{t+1}^2 = \mathbb{E}[(Z^{t+1})^2] = \mathbb{E}[f_t(Z^t)^2]$$

$$= \mathbb{E}[f_t(\sigma_t \cdot G)^2] \quad , \; G \sim N(0,1).$$

Proof technique: Gaussian conditioning.

Instead of asking about dist of x^{t+1} given past iterates, we ask about the dist of A given the past iterates.

$$1 \le t \le T-1: \quad A\, m^t = x^{t+1} + b_t\, m^{t-1} = y^t \quad \circledast$$

$$\mathcal{F}_t = \sigma(x^0, x^1, \dots, x^t).$$

$m^0, \dots, m^t$ are $\mathcal{F}_t$-measurable.
$b^0, \dots, b^t$ — — — ,

Conditional on $\mathcal{F}_T$ $\{m^t, y^t\}_{t=1}^{T}$ are known,

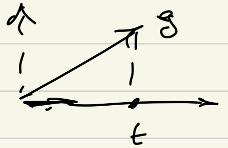
$A \mid \mathcal{F}_T$ is equivalent to conditioning A

on the linear system $\circledast$: $\quad A\, m^t = y^t$
$\forall\; 0 \le t \le T-1.$

Since A is gaussian :

$$A_{/\mathcal{F}_t} \overset{d}{=} \mathbb{E}[A \mid \mathcal{F}_t] + \mathcal{P}_t(\tilde{A}) \,, \quad \tilde{A} \overset{d}{=} A \ \text{(indep)}.$$

where $\mathcal{P}_t$ is the orthogonal projector onto the subspace $\{A \in \mathbb{R}^{n\times n} : A = A^T,\ A m^t = 0 \ \ \forall\, 0 \le t \le T-1\}$



$$g /_{g^T u = t} \overset{d}{=} t u + P_{u^\perp} \tilde{g} .$$

$\|u\| = 1$

write the AMP iteration :

$$x^{t+1}{}_{/\mathcal{F}_t} \overset{d}{=} A_{/\mathcal{F}_t} m^t - b_t\, m^{t-1}$$

$$= \mathbb{E}[A \mid \mathcal{F}_t]\, m^t + \mathcal{P}_t(\tilde{A})\, m^t - b_t \cdot m^{t-1}.$$

$$= \tilde{A} m^t + \underbrace{(\mathcal{P}_t(\tilde{A}) - \tilde{A})}\, m^t + \mathbb{E}[A \mid \mathcal{F}_t]\, m^t - b_t \cdot m^{t-1}.$$

$$= \underbrace{\tilde{A} m^t}_{\text{Gaussian}} - \underbrace{\mathcal{P}_t^{\perp}(\tilde{A})\, m^t}_{\| \ \ \| \to 0} + \underline{\qquad\qquad}$$

$\mathbb{E}[A \mid \mathcal{F}_t]\, m^t$ and $b_t \cdot m^{t-1}$ will nearly cancel out.

$\Rightarrow$ $x^{t+1} \mid \mathcal{F}_t$ is Gaussian centered.

by induction : $(x^1, \dots, x^t)$ is centered Gaussian

$$\frac{1}{n} \langle x^{t+1}, x^{s+1} \rangle \simeq \frac{1}{n} \langle \tilde{A} m^t, \tilde{A} m^s \rangle \simeq \frac{1}{n} \langle m^t, m^s \rangle$$

$$= \frac{1}{n} < f_t(x^t), f_s(x^s) >$$

- The actual proof: The Long AMP (LAMP)

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- $$\begin{cases} h^{t+1} = P_{t-1}^{\perp} A P_{t-1}^{\perp} q^t + \underbrace{H_{t-1} \alpha^t}_{} . \\ q^t = f_t(h^t) \end{cases} , \quad \sum_{s=0}^{t} \alpha_{s-1}^t h^s$$

init: $h^0 = x^0$, $h^1 = A q^0 = A f_0(h^0)$.

$$Q_{t-1} = [q^0 | \text{---} | q^{t-1}] \in \mathbb{R}^{n \times t}$$

$$P_{t-1} = \text{orthogonal proj onto Span}(q^0, \dots, q^t)$$

$$= Q_{t-1} (Q_{t-1}^T Q_{t-1})^{-1} Q_{t-1}^T .$$

$$H_{t-1} = [h^1 | \text{---} | h^t] \in \mathbb{R}^{n \times t} .$$

$$\alpha^t = \underset{\alpha \in \mathbb{R}^t}{\operatorname{argmin}} \| q^t - \sum_{s=0}^{t-1} \alpha_s^t q^s \|_2^2$$

i.e:

$$q_{\parallel}^t = \text{Proj of } q^t \text{ on Span}\{q^0, \dots, q^{t-1}\}$$

$$= P_{t-1} q^t = \sum_{s=0}^{t-1} \alpha_s^t q^s .$$

Proposition 1:

LAMP and AMP trajectories are close:

$$\begin{cases} \frac{1}{\sqrt{n}} \| h^{t+1} - x^{t+1} \|_2 \xrightarrow{P} 0 & \forall t \\ \frac{1}{\sqrt{n}} \| q^t - m^t \|_2 \xrightarrow{P} 0 & \forall t . \end{cases}$$

Proposition 2 : Geometry of iterates of LAMP:

1. $\forall\, t \geq 1$, $1 \leq r, s \leq t$:

$$\frac{1}{n} \langle h^{r+1}, h^{s+1} \rangle \overset{P}{\simeq} \frac{1}{n} \langle q^r, q^s \rangle .$$

$a_n \overset{P}{\simeq} b_n$ means $a_n - b_n \longrightarrow 0$ in prob.

2. $$\frac{1}{n} \sum_{i=1}^{n} \phi (h_i^0, \ldots, h_i^t) \xrightarrow[n \to +\infty]{P} \mathbb{E} \phi (Z^0, \ldots, Z^t)$$

$(Z^0, \ldots, Z^t)$ ~~has~~ the same ~~law~~ as in SE theorem.

Proposition 3 :

$$h^{t+1} |_{\mathfrak{F}_t} \overset{d}{=} P_{t-1}^{\perp} A^{new} P_{t-1}^{\perp} q^t + H_{t-1} \alpha^t .$$

$\mathfrak{F}_t = \sigma (h^0, \ldots, h^t)$. $A^{new} \overset{d}{=} A$, indep of A .

Proof of Prop 3 :

$$q_\perp^t = P_{t-1}^{\perp} q^t$$

P_{t-1} = Proj onto span$\{ q^0, \ldots, q^{t-1} \}$

$$\tilde{Q}_{t-1} = [q_\perp^0 | \ldots | q_\perp^{t-1}] .$$

$$\tilde{P}_{t-1} = \text{Proj onto span} \{ q_\perp^0, \ldots, q_\perp^{t-1} \}$$

$$= P_{t-1} .$$

$q_\perp^0 = q^0$

$$h^{t+1} = P_{t-1}^{\perp} A P_{t-1}^{\perp} q_{\perp}^{t} + H_{t-1}\alpha^{t}$$

$$= \tilde{P}_{t-1}^{\perp} A \tilde{P}_{t-1} q_{\perp}^{t} + H_{t-1}\alpha^{t}.$$

We have $\tilde{P}_{t-1} = \tilde{Q}_{t-1}(\tilde{Q}_{t-1}^{T}\tilde{Q}_{t-1})^{-1}\tilde{Q}_{t-1}$.

Consider the term $A\tilde{Q}_{t-1} = [A q_{\perp}^{s}]_{s=0}^{t-1}$ which shows up in $A\tilde{P}_{t-1}$.

Let $y^{t} = A q_{\perp}^{t}$.

- want to argue that y^{t-1} is $\mathcal{F}_{t}$-measurable.

$$y^{t} = A q_{\perp}^{t}$$

$$h^{t+1} = \tilde{P}_{t-1}^{\perp} A q_{\perp}^{t} + H_{t-1}\alpha^{t}$$

$$= (I - \tilde{P}_{t-1}) A q_{\perp}^{t} + H_{t-1}\alpha^{t}.$$

$$= y^{t} - \tilde{P}_{t-1} A q_{\perp}^{t} + H_{t-1}\alpha^{t}.$$

$$\Rightarrow y^{t} = h^{t+1} + \tilde{P}_{t-1} A q_{\perp}^{t} - H_{t-1}\alpha^{t}.$$

$$= h^{t+1} + \tilde{Q}_{t-1}(\tilde{Q}_{t-1}^{T}\tilde{Q}_{t-1})^{-1}\underbrace{\tilde{Q}_{t-1}^{T} A}_{= Y_{t-1}^{T}} q_{\perp}^{t} - H_{t-1}\alpha^{t}.$$

where we define

$$Y_{t-1} = [y^{1} | \ldots | y^{t-1}].$$

Therefore, by induction, y^t is $\mathcal{F}_{t+1}$-measurable.

$\Rightarrow$ conditionning A on the past i.e. $\mathcal{F}_t = \sigma(h^0, \dots, h^t)$

is equivalent to conditionning A on

$$A q_\perp^s = y^s \qquad 0 \leq s \leq t-1 .$$