Lec 27,28 addendum

Some of the properties I tried to convey, with suggested exercises:

Consider a system of three qubits A, B, C. The entanglements of A with B and A with C can be characterized by a quantity called the 'tangle'[†], which quantifies the extent to which A's entanglement with B limits its entanglement with C, and vice versa. In particular the tangle satisfies

$$\tau_{AB} + \tau_{BC} \le \tau_{A(BC)} \tag{1}$$

(where $\tau_{A(BC)}$ measures the entanglement of A with the combined BC system, as explained below).

To define the tangle for a two qubit system with density matrix ρ_{AB} (mixed or pure), we first construct the "spin-flipped" density matrix $\tilde{\rho}_{AB}$:

$$\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$$

(where ρ^* is the complex conjugate, and). Although not necessarily hermitian, the operator product $\rho_{AB}\tilde{\rho}_{AB}$ has real non-negative eigenvalues λ_i^2, $i = 1, 2, 3, 4$. Ordering their square roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ from largest to smallest, the tangle of the density matrix ρ_{AB} is defined as[††]

$$\tau_{AB} = [\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}]^2 \tag{2}$$

Some properties:

1) τ_{AB} ranges from 0 to 1, where $\tau_{AB} = 0$ means unentangled, and $\tau_{AB} = 1$ indicates maximally entangled.

2) if ρ_{AB} is a pure state, then only $\lambda_1 > 0$ and[*] $\tau_{AB} = \lambda_1^2 = 4 \det \rho_A$ (where $\rho_A = \mathrm{tr}_B \rho_{AB}$)

Examples from class (see p.4 for details):

a) show $\rho_{AB} = \mathrm{tr}_C |\psi_{GHZ}\rangle\langle\psi_{GHZ}|$ has $\tau_{AB} = 0$, where $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big)_{ABC}$

b) show $\rho_{AB} = |\psi_{11}\rangle\langle\psi_{11}|$ has $\tau_{AB} = 1$, where $|\psi_{11}\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)_{AB}$

c) more generally calculate τ_{AB} for $\rho_{AB} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \alpha_0|00\rangle + \alpha_1|11\rangle$

---

† Coffman, Kundu, Wootters (1999), https://arxiv.org/abs/quant-ph/9907047

†† The definition is closely related to prior work on "entanglement of formation", and is the square of a quantity known in that context as the "concurrence".

* Start from a general $|\psi\rangle = \sum_{ij} \alpha_{ij}|i\rangle|j\rangle$, use $i(\sigma_y)_{ij} = \epsilon_{ij}$ (where $\epsilon_{01} = -\epsilon_{10} = 1$, $\epsilon_{ii} = 0$) satisfying $\epsilon_{ij}\epsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$, and for a $2{\times}2$ matrix A recall $\epsilon_{ik}\epsilon_{jl}A_{ij}A_{kl} = 2 \det A$. (See derivation I on p.3)

Suppose now that ρ_{ABC} describes a pure state of 3 qubits. The formula for the tangle simplifies since any pair of qubits is entangled with only one other qubit, a two-state system, and the reduced density matrix of any pair thus has only two non-zero eigenvalues. The product $\rho_{AB}\tilde{\rho}_{AB}$ also has only two non-zero eigenvalues, so from the earlier definition

$$\tau_{AB} = (\lambda_1 - \lambda_2)^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 = \mathrm{tr}(\rho_{AB}\tilde{\rho}_{AB}) - 2\lambda_1\lambda_2 \leq \mathrm{tr}(\rho_{AB}\tilde{\rho}_{AB}) \ .$$

It is straightforward* to write the right hand side above as

$$\mathrm{tr}(\rho_{AB}\tilde{\rho}_{AB}) = 2(\det\rho_A + \det\rho_B - \det\rho_C) \ ,$$

where ρ_A, ρ_B, ρ_C are the reduced 1-qubit density matrices. By symmetry in B and C, it follows that

$$\mathrm{tr}(\rho_{AB}\tilde{\rho}_{AB}) + \mathrm{tr}(\rho_{AC}\tilde{\rho}_{AC}) = 4\det\rho_A$$

establishing the advertised identity (1):

$$\tau_{AB} + \tau_{AC} \leq \tau_{A(BC)}$$

(where $\tau_{A(BC)} = 4\det\rho_A$, since the ABC system is assumed to be a pure state). The entanglement of A with the pair BC bounds A's entanglement with B,C individually, and the bound on their sum quantifies how any entanglement devoted to B is not available to C and vice versa.

Comments:

1) the inequality is saturated by states of the form $|\psi\rangle = \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle$ (Show for this state that $\tau_{AB} = 4|\alpha|^2|\beta|^2$, $\tau_{AC} = 4|\alpha|^2|\gamma|^2$, and $\tau_{A(BC)} = 4|\alpha|^2(|\beta|^2 + |\gamma|^2)$, by calculating ρ_{AB}, $\tilde{\rho}_{AB}$, and determining τ_{AB} from the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$, then by calculating $\tau_{A(BC)} = 4\det\rho_A$, where $\rho_A = \mathrm{tr}_B\rho_{AB}$.)

2) The generalization to n qubits was proven six years later‡

$$\sum_{k=2}^{n} \tau(\rho_{A_1 A_k}) \leq \tau(\rho_{A_1(A_2 \ldots A_n)})$$

showing that the entanglement of any qubit with the rest together bounds the sum of its entanglements with each of the other qubits individually, where the entanglement remains quantified by the tangle τ defined in eqn. (2).

---

* Start from a general $|\psi\rangle = \sum_{ijk} \alpha_{ijk}|i\rangle|j\rangle|k\rangle$, use same relations as in last footnote previous page (see derivation II on p.3)

‡ Osborne,Verstraete (2005), https://arxiv.org/abs/quant-ph/0502176

For ease of notation, the derivations below use a standard convention in which repeated indices are assumed to be summed over, e.g., $\alpha_{ij}\alpha^*_{kj}$ represents $\sum_{j=0}^{1}\alpha_{ij}\alpha^*_{kj} = (\alpha\alpha^\dagger)_{ik}$.

I. The 2-qubit state $|\psi\rangle_{AB} = \sum_{ij}\alpha_{ij}|i\rangle|j\rangle$, has pure density matrix

$$\rho_{AB} = \alpha_{ij}|i\rangle|j\rangle\langle k|\langle \ell|\alpha^*_{k\ell} \ ,$$

and reduced density matrix $\rho_A = \mathrm{tr}_B\rho_{AB} = \alpha_{ij}|i\rangle\langle k|\alpha^*_{kj} = (\alpha\alpha^\dagger)_{ik}|i\rangle\langle k|$, so $\rho_A = \alpha\alpha^\dagger$. We calculate $\mathrm{tr}\rho_{AB}\tilde\rho_{AB} = \alpha_{ij}\alpha^*_{k\ell}\epsilon_{kk'}\epsilon_{\ell\ell'}\alpha^*_{k'\ell'}\alpha_{i'j'}\epsilon_{i'i}\epsilon_{j'j}$ using $\epsilon_{\ell\ell'}\epsilon_{j'j} = \delta_{\ell j'}\delta_{\ell' j} - \delta_{\ell j}\delta_{\ell' j'}$ to give

$$\begin{aligned}\mathrm{tr}\rho_{AB}\tilde\rho_{AB} &= \alpha_{ij}\alpha^*_{k\ell}\epsilon_{kk'}\alpha^*_{k'j}\alpha_{i'\ell}\epsilon_{i'i} - \alpha_{ij}\alpha^*_{kj}\epsilon_{kk'}\alpha^*_{k'j'}\alpha_{i'j'}\epsilon_{i'i}\\ &= \epsilon_{kk'}\epsilon_{i'i}(\alpha\alpha^\dagger)_{ik'}(\alpha\alpha^\dagger)_{i'k} - \epsilon_{kk'}\epsilon_{i'i}(\alpha\alpha^\dagger)_{ik}(\alpha\alpha^\dagger)_{i'k'} = 2\det\alpha\alpha^\dagger + 2\det\alpha\alpha^\dagger = 4\det\rho_A\end{aligned}$$

II. The 3 qubit state $|\psi\rangle = \sum_{ijk}\alpha_{ijm}|i\rangle|j\rangle|m\rangle$ has pure density matrix

$$\rho_{ABC} = \alpha_{ijk}|i\rangle|j\rangle|m\rangle\langle k|\langle\ell|\langle n|\alpha^*_{k\ell n} \ ,$$

and reduced density matrix $\rho_{AB} = \mathrm{tr}_C\rho_{ABC} = \alpha_{ijm}\alpha^*_{k\ell m}|i\rangle|j\rangle\langle k|\langle\ell|$.
Then $\rho_A = \mathrm{tr}_B\rho_{AB} = \alpha_{ikm}\alpha^*_{jkm}|i\rangle\langle j|$, similarly $\rho_B = \mathrm{tr}_A\rho_{AB} = \alpha_{kim}\alpha^*_{kjm}|i\rangle\langle j|$, and $\rho_C = \mathrm{tr}_A\mathrm{tr}_B\rho_{ABC} = \alpha_{kmi}\alpha^*_{kmj}|i\rangle\langle j|$. It follows that

$$\mathrm{tr}\rho_{AB}\tilde\rho_{AB} = \alpha_{ijm}\alpha^*_{k\ell m}\epsilon_{kk'}\epsilon_{\ell\ell'}\alpha^*_{k'\ell'm'}\alpha_{i'j'm'}\epsilon_{i'i}\epsilon_{j'j} \ .$$

As above, using $\epsilon_{\ell\ell'}\epsilon_{j'j} = \delta_{\ell j'}\delta_{\ell' j} - \delta_{\ell j}\delta_{\ell' j'}$ gives

$$\mathrm{tr}\rho_{AB}\tilde\rho_{AB} = \alpha_{ijm}\alpha^*_{k\ell m}\epsilon_{kk'}\alpha^*_{k'jm'}\alpha_{i'\ell m'}\epsilon_{i'i} - \alpha_{ijm}\alpha^*_{kjm}\epsilon_{kk'}\alpha^*_{k'j'm'}\alpha_{i'j'm'}\epsilon_{i'i} \ .$$

Now in the first term, expand as well $\epsilon_{kk'}\epsilon_{i'i} = \delta_{ki'}\delta_{k'i} - \delta_{ki}\delta_{k'i'}$, giving

$$\begin{aligned}\mathrm{tr}\rho_{AB}\tilde\rho_{AB} &= \alpha_{ijm}\alpha^*_{k\ell m}\alpha^*_{ijm'}\alpha_{k\ell m'} - \alpha_{ijm}\alpha^*_{i\ell m}\alpha^*_{i'jm'}\alpha_{i'\ell m'} - \epsilon_{kk'}\epsilon_{i'i}(\rho_A)_{ik}(\rho_A)_{i'k'}\\ &= (\rho_C)_{mm'}(\rho_C)_{m'm} - (\rho_B)_{j\ell}(\rho_B)_{\ell j} + 2\det\rho_A\\ &= \mathrm{tr}(\rho_C^2) - \mathrm{tr}(\rho_B^2) + 2\det\rho_A\\ &= 2(\det\rho_A + \det\rho_B - \det\rho_C)\end{aligned}$$

(where the last line uses that a 2×2 matrix ρ with trace 1 has eigenvalues ξ, $(1-\xi)$ and therefore satisfies $\mathrm{tr}\rho^2 = \xi^2 + (1-\xi)^2 = -2\xi(1-\xi) + 1 = -2\det\rho + 1$).

Examples from class:

a) show $\rho_{AB} = \mathrm{tr}_C |\psi_{GHZ}\rangle\langle\psi_{GHZ}|$ has $\tau_{AB} = 0$, where $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big)_{ABC}$. Then $\rho_{AB} = \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big) = \tilde{\rho}_{AB}$ and

$$\rho_{AB}\tilde{\rho}_{AB} = \frac{1}{4}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

We see that $\lambda_1^2 = \lambda_2^2 = \frac{1}{4}$, $\lambda_3 = \lambda_4 = 0$, so $\tau = (\lambda_1 - \lambda_2)^2 = 0$.

b) show $\rho_{AB} = |\psi_{11}\rangle\langle\psi_{11}|$ has $\tau_{AB} = 1$, where $|\psi_{11}\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)_{AB}$. Then $\rho_{AB} = \frac{1}{2}\big(|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|\big) = \tilde{\rho}_{AB}$ and

$$\rho_{AB}\tilde{\rho}_{AB} = \frac{1}{2}\big(|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01\big) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

We see that $\lambda_1^2 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, so $\tau = \lambda_1^2 = 1$.

c) more generally calculate τ_{AB} for $\rho_{AB} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \alpha_0|00\rangle + \alpha_1|11\rangle$. Then $\rho_{AB} = |\alpha_0^2||00\rangle\langle 00| + \alpha_0\alpha_1^*|00\rangle\langle 11| + \alpha_0^*\alpha_1|11\rangle\langle 00| + |\alpha_1^2||11\rangle\langle 11|$ and $\tilde{\rho}_{AB} = |\alpha_1^2||00\rangle\langle 00| + \alpha_0\alpha_1^*|00\rangle\langle 11| + \alpha_0^*\alpha_1|11\rangle\langle 00| + |\alpha_0^2||11\rangle\langle 11|$, so that

$$\rho_{AB}\tilde{\rho}_{AB} = 2\begin{pmatrix} |\alpha_0^2\alpha_1^2| & 0 & 0 & |\alpha_0^2|\alpha_0\alpha_1^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ |\alpha_1^2|\alpha_0^*\alpha_1 & 0 & 0 & |\alpha_0^2\alpha_1^2| \end{pmatrix} .$$

We see that $\lambda_1^2 = 4|\alpha_0^2\alpha_1^2|$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, so $\tau = \lambda_1^2 = 4|\alpha_0^2\alpha_1^2|$. Note that the maximum $\tau = 1$ occurs when $|\alpha_0| = |\alpha_1| = 1/\sqrt{2}$, and τ goes to zero as either $|\alpha_1|$ or $|\alpha_0|$ goes to zero (i.e., in the limit of unentangled states $|00\rangle$ and $|11\rangle$).