$$\underbrace{V\left(\frac{1+(-1)^x V}{2}\right)}_{P_x^V} = (-1)^x \left(\frac{1+(-1)^x V}{2}\right)$$

Specify 5 qubit code

$$M_0 = Z X X Z I \qquad \underline{M_i^2 = 1}$$

$$M_1 = X X Z I Z$$

$$M_2 = X Z I Z X$$

$$M_3 = Z I Z X X$$

$$[X_i, Z_j] = 0 \quad i \neq j$$

Commutator

$[A, B] = AB - BA$

$$[M_i, M_j] = 0$$

$$M_4 = I\,Z\,X\,X\,Z \;?$$

not independent $= M_0 M_1 M_2 M_3$

Code words

$$|\bar{0}\rangle = \frac{1}{4}(1+M_0)(1+M_1)(1+M_2)(1+M_3)|00000\rangle$$

$$|\bar{1}\rangle = \frac{1}{4}(1+M_0)(1+M_1)(1+M_2)(1+M_3)|11111\rangle$$

Normalized? $\quad (1+M_i)^2 = 2(1+M_i)$

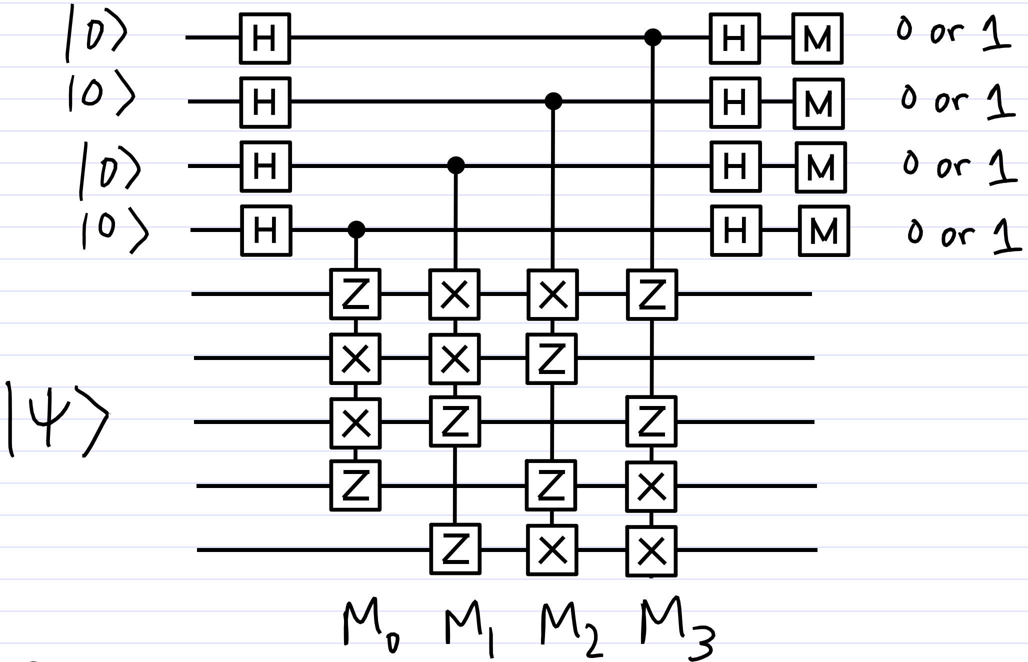
$$\langle\bar{0}|\bar{0}\rangle = \frac{1}{16}16\langle 0^5|\prod_i(1+M_i)|0^5\rangle = \underline{1}$$

$$\langle\bar{1}|\bar{1}\rangle = \underline{1} \qquad \bar{X} = X_0X_1X_2X_3X_4$$

$$\langle\bar{1}|\bar{0}\rangle = \langle\bar{0}|\bar{1}\rangle = 0 \qquad \bar{Z} = Z_0Z_1Z_2Z_3Z_4$$

$$|\bar{1}\rangle = \bar{X}|\bar{0}\rangle \qquad \bar{Z}|\bar{0}\rangle = |\bar{0}\rangle \qquad [\bar{X}, M_i] = 0$$

$$|\bar{0}\rangle = \bar{X}|\bar{1}\rangle \qquad \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle \qquad [\bar{Z}, M_i] = 0$$



5-qubit code, encoding circuit.

To initialize state to $|\bar{0}\rangle$:

measure M_i's, projects to

$\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$ space.

measure $\bar{Z}$, gives $|\bar{0}\rangle$ or $|\bar{1}\rangle$

if $|\bar{1}\rangle$, apply $\bar{X}|\bar{1}\rangle = |\bar{0}\rangle$

Next: We'll show that any of the fifteen

$X_i, Y_i, Z_i \quad i = 0, \ldots, 4$

has a unique signature in eigenvalues of the M_i.

e.g. $|\psi\rangle \to X_3 |\psi\rangle$, then since X_3 commutes with M_0, M_1, M_3, and anti-commutes with M_2, only M_2 flips sign, and the M_i would be measured as

$$(+, +, -, +)$$

$M_0 = Z_1 X_2 X_3 Z_4$ $\qquad$ $M_2 = Z_3 X_4 X_0 Z_1$

$M_1 = Z_2 X_3 X_4 Z_0$ $\qquad$ $M_3 = Z_4 X_0 X_1 Z_2$

Now see that the M_i characterize the 16 spaces $\underbrace{\mathbb{1}}_{\text{uncorrupted}}$ $\underbrace{X_i\ Y_i\ Z_i}_{\text{15 corruptions}}$:

	X_0	Y_0	Z_0	X_1	Y_1	Z_1	X_2	Y_2	Z_2	X_3	Y_3	Z_3	X_4	Y_4	Z_4	$\mathbb{1}$
M_0	+	+	+	−	−	+	+	−	−	+	−	−	−	−	+	+
M_1	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+
M_2	+	−	−	−	−	+	+	+	+	−	−	+	+	−	−	+
M_3	+	−	−	+	−	−	−	−	+	+	+	+	−	−	+	+

each column is a unique error signature. Just look at whether the given operator commutes or anti-commutes with M_i.

e.g., (start of 1st column)

$$M_0 X_0 |\psi\rangle = X_0 M_0 |\psi\rangle = +X_0|\psi\rangle$$

$$M_1 X_0 |\psi\rangle = -X_0 M_1 |\psi\rangle = -X_0|\psi\rangle$$

Recall $|\bar{0}\rangle = \frac{1}{4} \prod_i (1+M_i) |0^5\rangle$

$$|\bar{1}\rangle = \frac{1}{4} \prod_i (1+M_i) |1^5\rangle$$

have $M_0, M_1, M_2, M_3 = +1, +1, +1, +1$

Suppose: measure M_0, M_1, M_2, M_3

as $+1, -1, +1, -1$

How to correct error?

Well $+-+-$ is the X_2 column,

so the state has an X_2 error

$$X_2 |\psi\rangle$$

To correct, apply X_2

$$X_2 X_2 |\psi\rangle = |\psi\rangle$$

How to implement arbitrary U?

- Universal gate set

  CNOT, Z, H, T $\qquad T^4 = Z$ ($\pi/8$ gate)

  gives any U

difficulty with 5-qubit code:

"hard" to get H, CNOT

instead use:

7-qubit "Steane code"

7-qubit code

$$M_0 = X_0 X_4 X_5 X_6 \qquad N_0 = Z_0 Z_4 Z_5 Z_6$$
$$M_1 = X_1 X_3 X_5 X_6 \qquad N_1 = Z_1 Z_3 Z_5 Z_6$$
$$M_2 = X_2 X_3 X_4 X_6 \qquad N_2 = Z_2 Z_3 Z_4 Z_6$$

$$M_i^2 = 1 = N_i^2 \qquad [M_i, M_j] = [N_i, N_j] = 0$$
$$[M_i, N_j] = 0$$

$$|\bar{0}\rangle = \frac{1}{2^{3/2}}(1+M_0)(1+M_1)(1+M_2)|0^7\rangle$$

$$|\bar{1}\rangle = \frac{1}{2^{3/2}}\prod_{i=0}^{2}(1+M_i)|1^7\rangle$$

$$\langle\bar{0}|\bar{0}\rangle = \frac{1}{2^3}2^3\langle 0^7|\prod_i(1+M_i)|0^7\rangle$$

$(1+M_i)^2 = 2(1+M_i)$

(only contribution)

same for:

$$\langle\bar{1}|\bar{1}\rangle = 1$$

$$\langle\bar{0}|\bar{1}\rangle = \langle\bar{1}|\bar{0}\rangle = 0$$

(odd/even)

$\bar{X} = X_0 X_1 \ldots X_6$

$\bar{Z} = Z_0 Z_1 \ldots Z_6$

$\bar{X}|\bar{0}\rangle = \bar{1} \quad \bar{X}|\bar{1}\rangle = |\bar{0}\rangle$

$\bar{Z}|\bar{0}\rangle = |\bar{0}\rangle \quad \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle$

$\bar{X}^2 = \bar{Z}^2 = 1 \quad \bar{X}\bar{Z} = -\bar{Z}\bar{X}$.

Now $\bar{H} = H_0 H_1 H_2 H_3 H_4 H_5 H_6 = H^{\otimes 7}$

need to show:

$\bar{H}|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|\bar{0}\rangle + |\bar{1}\rangle) \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\bar{H}|\bar{1}\rangle = \frac{1}{\sqrt{2}}(|\bar{0}\rangle - |\bar{1}\rangle)$

$\langle\bar{0}|\bar{H}|\bar{0}\rangle = \langle\bar{0}|\bar{H}|\bar{1}\rangle = \langle\bar{1}|\bar{H}|\bar{0}\rangle$

$= -\langle\bar{1}|\bar{H}|\bar{1}\rangle = 1/\sqrt{2}$