Classical Error Correction

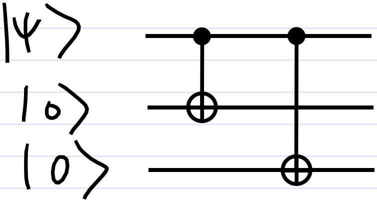
$$|\bar{0}\rangle = |0\rangle|0\rangle|0\rangle = |000\rangle$$

$$|\bar{1}\rangle = |1\rangle|1\rangle|1\rangle = |111\rangle$$

$$|0\rangle|1\rangle|0\rangle \quad \begin{matrix} \boxed{|\bar{0}\rangle} \\ |\bar{1}\rangle \end{matrix} \; ?$$ majority rule

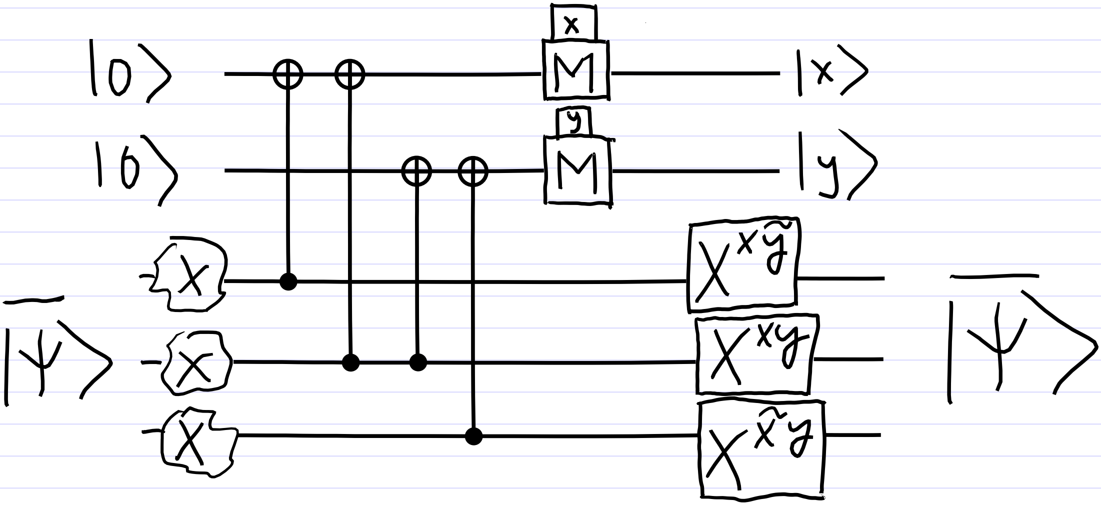
Quantum mechanical!

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



$$\alpha|000\rangle + \beta|111\rangle$$

$$= |\bar{\psi}\rangle$$



restores state $|\bar{\psi}\rangle$

Learn nothing about α, β,

only relations within codewords

xy	
00	$\underline{1}$
10	X_2
01	X_0
11	X_1

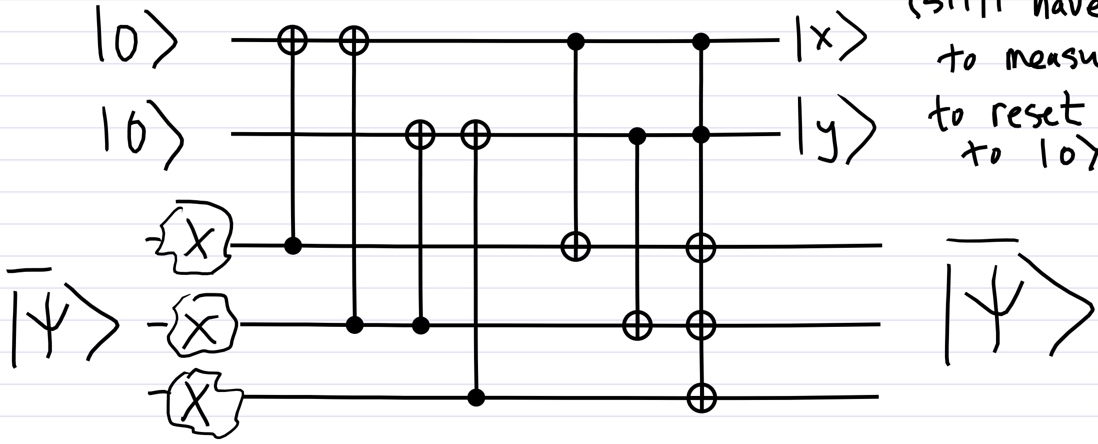
#dims #possible corruptions

$$2(3+1) = 2^3$$

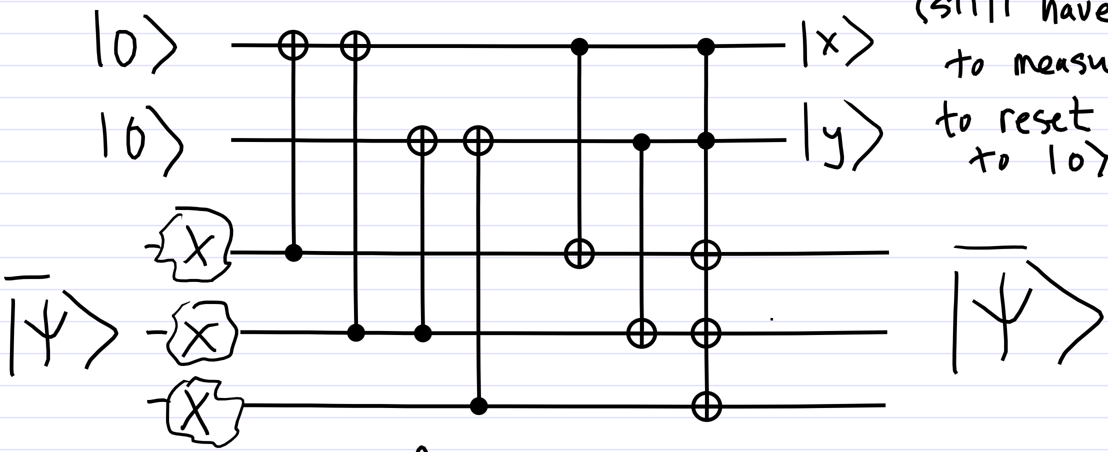
#qubits

$$2(n+1) \leq 2^n \qquad n \geq 3$$

3 qubit codeword is minimum for correcting single bitflip error



(still have to measure to reset to $|0\rangle$)



(still have to measure to reset to $|0\rangle$)

For generalization, consider eigenvalues of $Z_1 Z_2$, $Z_0 Z_1$

Error	$Z_1 Z_2$	$Z_0 Z_1$
$\mathbb{1}$	1	1
X_2	-1	1
X_1	-1	-1
X_0	1	-1
	$(-1)^x$	$(-1)^y$

e.g. $(Z_0 Z_1) X_0 \overline{|\psi\rangle} = -X_0 (Z_0 Z_1) \overline{|\psi\rangle} = -X_0 \overline{|\psi\rangle}$

$$\boxed{(Z_1 Z_2)\, X_2|\overline{\psi}\rangle \overset{?}{=} \pm X_2|\overline{\psi}\rangle}$$

$$|\overline{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$$

$$Z_0 Z_1|\overline{\psi}\rangle = |\overline{\psi}\rangle$$

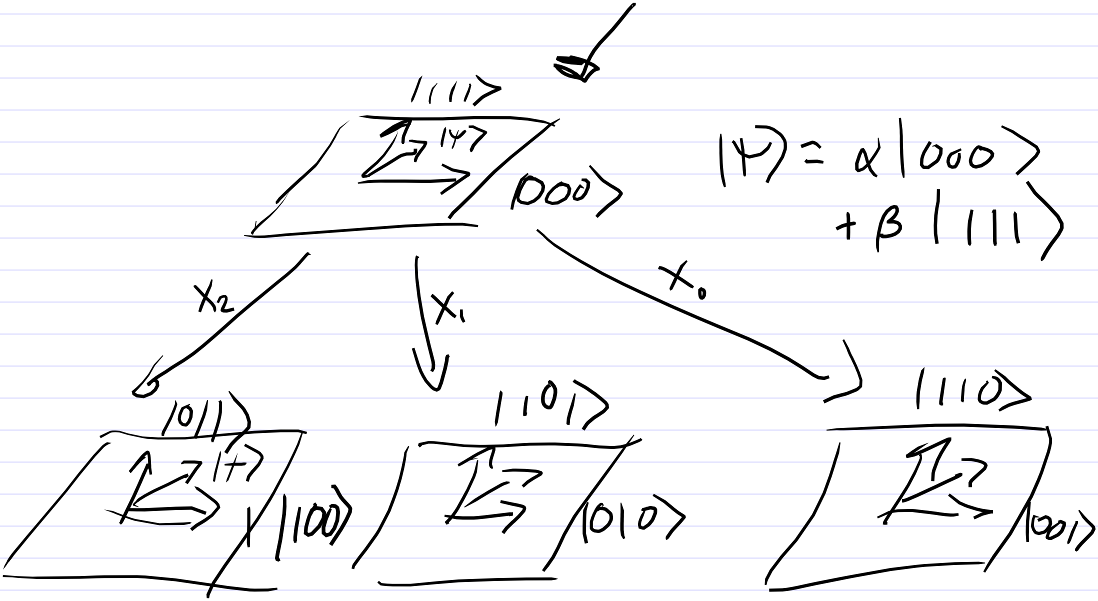
$$Z_1 Z_2|\overline{\psi}\rangle = |\overline{\psi}\rangle$$

$$(Z_1 Z_2 \quad X_2) = -X_2\,(Z_1 Z_2)$$

$$(Z \otimes Z \otimes \mathbb{1} \qquad X \otimes 1 \otimes \underline{1}$$

$$\underline{(Z_0 Z_1)}\, X_2|\overline{\psi}\rangle \overset{?}{=} \pm X_2|\overline{\psi}\rangle$$

$(Z_0 Z_1)^2 = 1$ so has eigenvalues ± 1



$$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$$

"Stabilizer formalism"

"Measure" an operator

$A^2 = 1$, A hermitian

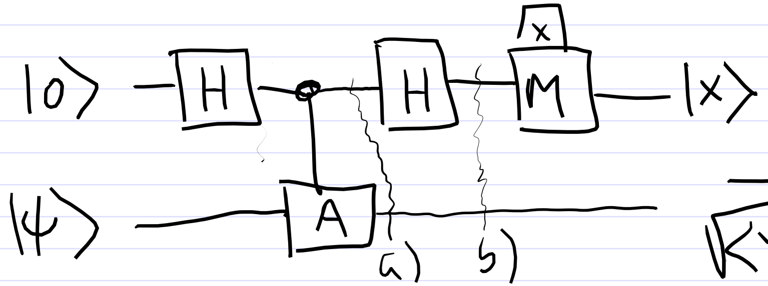
$\Rightarrow$ eigenvalues $= \pm 1$

$$A = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & \ddots \end{pmatrix} \qquad P_0^A = \frac{1+A}{2} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$P_1^A = \frac{1-A}{2} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$

Projectors onto ±1 eigenspaces.

eigenvalue of P_x^A is $(-1)^x$



$$\frac{P_x^A|\psi\rangle}{\sqrt{\langle\psi|P_x^A|\psi\rangle}}$$

a) $C^A \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle+|1\rangle A|\psi\rangle\right)$

b) $\frac{1}{2}\left((|0\rangle+|1\rangle)|\psi\rangle + (|0\rangle-|1\rangle)A|\psi\rangle\right)$

$$= \frac{1}{2}|0\rangle(1+A)|\psi\rangle+\frac{1}{2}|1\rangle(1-A)|\psi\rangle$$

$$= |0\rangle P_0^A|\psi\rangle + |1\rangle P_1^A|\psi\rangle$$

Note: if $A = Z$ for single qubit, coincides with usual notion of measurement

Now consider more than just bitflips:

$$|\Psi_0\rangle = \alpha|0\rangle|v\rangle_n + \beta|1\rangle|w\rangle_n$$

$$X|\Psi_0\rangle = |\Psi_1\rangle = \alpha|1\rangle|v\rangle_n + \beta|0\rangle|w\rangle_n$$

$$Z|\Psi_0\rangle = |\Psi_2\rangle = \alpha|0\rangle|v\rangle_n - \beta|1\rangle|w\rangle_n$$

$$XZ|\Psi_0\rangle = |\Psi_3\rangle = \alpha|1\rangle|v\rangle_n - \beta|0\rangle|w\rangle_n$$

These form a basis for the full 2d (complex) space of all corruptions.

$\Longrightarrow$ Any corruption can be expanded in terms of X, Z, XZ

(bit flip, phase error, joint bitflip/phase error ~Y)

If can correct these, can correct all!

n qubit error-correcting code

n = # qubits, low error rate, so only single qubit errors:

$$|\psi\rangle \rightarrow \left(\mathbb{1} + \sum_{j=0}^{n-1}\left(\varepsilon_x^j X_j + \varepsilon_y^j Y_j + \varepsilon_z^j Z_j\right)\right)|\psi\rangle$$

(Any of single X, Y, Z error on any of n-qubits in codeword)

To have orthogonal subspaces to correct any of $3n+1$ errors, need

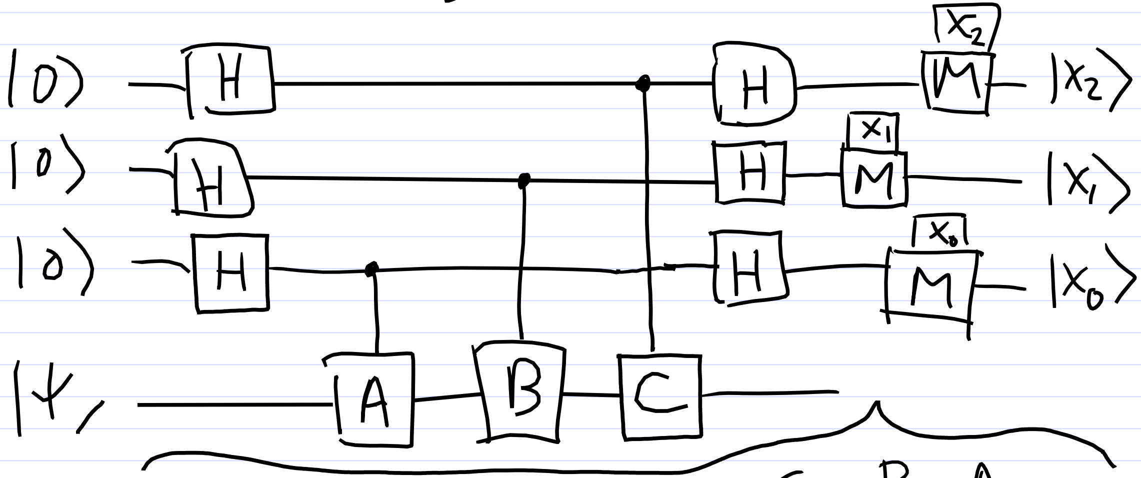
$$2(3\underset{\overset{\uparrow}{X_i,\,Y_i,\,Z_i}}{n}+1) \le 2^n$$

i.e., $n \geq 5$

and there will be a minimal $n=5$ code that corrects all three types of error.

Multiple operator measurements

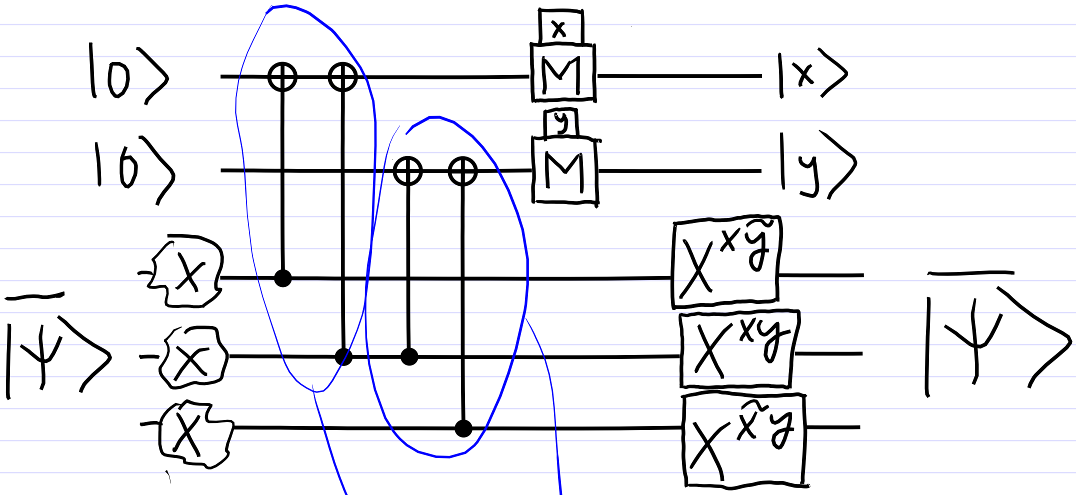
$$A^2 = B^2 = C^2 = 1$$



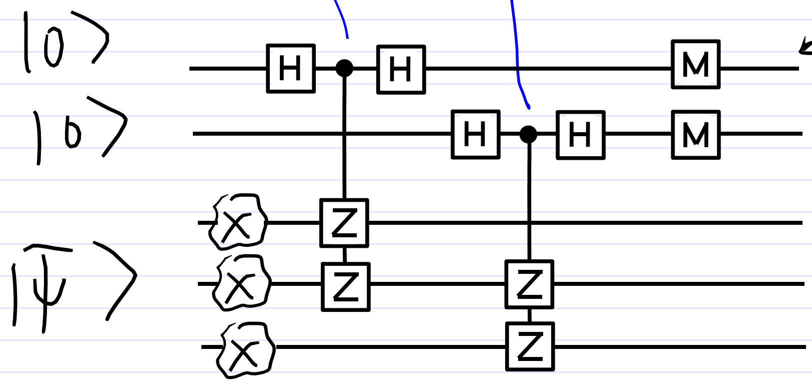
$$\sum_{x_2,x_1,x_0} |x_2\rangle |x_1\rangle |x_0\rangle P^C_{x_2} P^B_{x_1} P^A_{x_0} |\psi\rangle$$

$\Rightarrow$ If A, B, C are mutually commuting, then measuring X_0, X_1, X_2 projects onto their joint eigenspaces.

±1



Use $X = HZH$



"measures" $Z_1 Z_2$

measures $Z_0 Z_1$