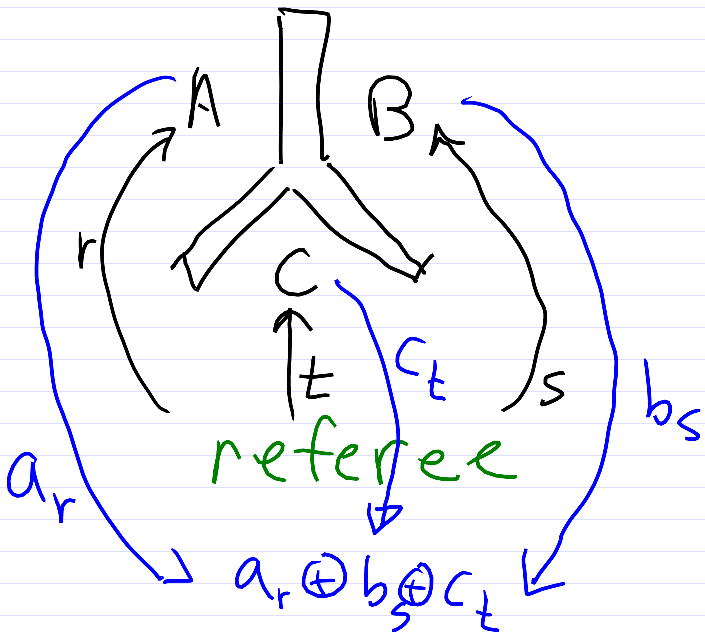
Physics		"CS"
experiment (to understand nature)	$\Leftrightarrow$	games: optimize performance in various settings
locality	$\Leftrightarrow$	no communication
hidden variable can't explain statistics	$\Leftrightarrow$	classical strategy can't do as well as quantum

GHZ game

A, B, C are given r, s, t
from {000, 011, 110, 101}

goal: a, b, c $\quad a \oplus b \oplus c = r \vee s \vee t$



if A receives $r=0 \Rightarrow a_0$

$r=1 \quad a_1$

" B " $s=0,1 \quad b_0, b_1$

" C " $t=0,1 \quad c_0, c_1$

r, s, t	$a \oplus b \oplus c$	$r \vee s \vee t$
0 0 0	$a_0 \oplus b_0 \oplus c_0 =$	0
0 1 1	$a_0 \oplus b_1 \oplus c_1 =$	1
1 1 0	$a_1 \oplus b_1 \oplus c_0 =$	1
1 0 1	$a_1 \oplus b_0 \oplus c_1 =$	1
	$0 \oplus 0 \oplus 0 =$	1

inconsistent! can only solve 3 of 4, so best is 75%

QM A, B, C share

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

if receive $r, s, t =$ 0, measure in X basis

" " " 1 " " Y "

measure in X basis $\iff$ apply H

" " Y " $\iff$ " $HS^\dagger$

$$SH|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = |+i\rangle$$

$$S = \begin{pmatrix} 1 & \\ & +i \end{pmatrix} \qquad SH|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$$= |-i\rangle$$

$|\pm i\rangle$ are eigenstates of $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\text{Define}\quad \left.\begin{aligned} |\varphi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\ |\psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \end{aligned}\right\} \text{"Bell Basis"}$$

$$\text{Note}\quad \begin{aligned} H\otimes H\,|\varphi^{+}\rangle &= |\varphi^{+}\rangle \\ H\otimes H\,|\varphi^{-}\rangle &= |\psi^{+}\rangle \end{aligned}$$

$$\text{and}\quad \begin{aligned} (HS^{\dagger})\otimes(HS^{\dagger})\,|\varphi^{+}\rangle &= |\psi^{+}\rangle \\ (HS^{\dagger})\otimes(HS^{\dagger})\,|\varphi^{-}\rangle &= |\varphi^{+}\rangle \end{aligned}$$

$$H\otimes H\otimes H\,|\psi_{GHZ}\rangle = \frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right)$$

even # 1's

$r,s,t = 0,0,0$

Always win!

$$H\otimes HS^{\dagger}\otimes HS^{\dagger}\,|\psi_{GHZ}\rangle = \frac{1}{2}\left(|001\rangle + |010\rangle + |100\rangle + |111\rangle\right)$$

odd # 1's

$r,s,t \in \{0,1,1;\ 1,0,1;\ 1,1,0\}$

$$H \otimes 1 \otimes 1 \, |\psi_{GHZ}\rangle$$

$$= \frac{1}{2}\Big(\big(|0\rangle + |1\rangle\big)|0\rangle|0\rangle + \big(|0\rangle - |1\rangle\big)|1\rangle|1\rangle \Big)$$

$$= \frac{1}{\sqrt{2}}\Big(|0\rangle \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) + |1\rangle \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big) \Big)$$

$$= \frac{1}{\sqrt{2}}\Big(|0\rangle|\varphi^+\rangle + |1\rangle|\varphi^-\rangle \Big)$$

$$H \otimes H \, |\varphi^\pm\rangle = \frac{1}{2\sqrt{2}}\Big[\big(|0\rangle + |1\rangle\big)\big(|0\rangle + |1\rangle\big) \pm \big(|0\rangle - |1\rangle\big)\big(|0\rangle - |1\rangle\big) \Big]$$

$$= |\varphi^+\rangle$$

$$H \otimes H \, |\varphi^-\rangle = |\psi^+\rangle$$

$$(HS^\dagger \otimes HS^\dagger)\, |\varphi^+\rangle = |\psi^+\rangle$$

$$\frac{1}{2\sqrt{2}}\Big[\big(|0\rangle + |1\rangle\big)\big(|0\rangle + |1\rangle\big) \pm (-i)\big(|0\rangle - |1\rangle\big)(-i)\big(|0\rangle - |1\rangle\big) \Big]$$

$$(HS^\dagger \otimes HS^\dagger)\, |\varphi^-\rangle = |\varphi^+\rangle$$

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\varphi^+\rangle + |1\rangle|\varphi^-\rangle\right)$$

$$H\otimes H\otimes H|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\varphi^+\rangle + |1\rangle|\psi^+\rangle\right)$$

$$= \frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right)$$

even # 1's

$$H\otimes(HS^\dagger)\otimes(HS^\dagger)|\psi_{GHZ}\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|\psi^+\rangle + |1\rangle|\varphi^+\rangle\right)$$

$$= \frac{1}{2}\left(|001\rangle + |010\rangle + |100\rangle + |111\rangle\right)$$

odd # 1's !

Notice if we measure $H \otimes H \otimes H |\Psi\rangle$

x_2, x_1, x_0 satisfy $\boxed{x_0 \oplus x_1 \oplus x_2 = 0.}$

Therefore $x_0 = x_1 \oplus x_2$

But Einstein would expect

<u>same</u> x_0

regardless of HSt on 1,2:

① This was with H not $HS^\dagger$ on 1,2

② but measuring 1,2 doesn't affect 0

③ So X_0 must be "predisposed" even before measuring

④ & moreover even before deciding whether to apply H or $HS^\dagger$

The same argument applies to X_1, X_2.
And if X_i^s is the result of measuring after applying $HS^\dagger$ to qubit i, the X_i^s are also independent of what is (or is not) done to other qubits.
But those "predispositions" are not mutually consistent:

Experimentally, they must satisfy

$$\left\{\begin{array}{l} X_0 \oplus X_1^s \oplus X_2^s = 1 \\ X_0^s \oplus X_1 \oplus X_2^s = 1 \\ X_0^s \oplus X_1^s \oplus X_2 = 1 \end{array}\right.$$

with sum: $X_0 \oplus X_1 \oplus X_2 = 1$

so it is not possible to preassign values to X_0, X_1, X_2 consistent with all four measurement possibilities, since with all H's,

$$X_0 \oplus X_1 \oplus X_2 = 0.$$

and Einstein's "elements of reality"
[hidden variable] argument
says they're the <u>same</u> X_i,
regardless of $HS^\dagger$ applied
to other two.

Do the experiment:

w/ two $HS^\dagger$ always an odd # of 1s.

w/ three H always an even # of 1s.

So there are <u>no</u> consistent
pre-assignments/predispositions/
hidden variables / elements of reality

~~Einstein~~ ✗ QM ✓

m marked states

$$V|x\rangle = (-1)^{f(x)}|x\rangle$$

$$f(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$$

$Y =$ set of marked states

$$|Y| = m$$

$$|\varphi\rangle = \frac{1}{2^{n/2}} \sum_{0 \le x < 2^n} |x\rangle = \cos\theta |no\rangle + \sin\theta |yes\rangle$$

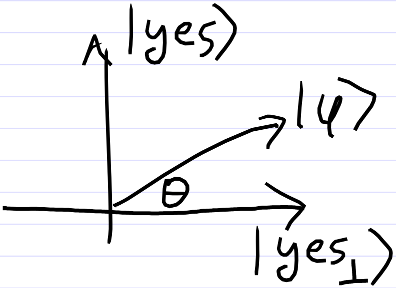
$$|yes\rangle = \frac{1}{\sqrt{m}} \sum_{x | f(x)=1} |x\rangle \qquad |no\rangle = \frac{1}{\sqrt{2^n - m}} \sum_{x | f(x)=0} |x\rangle$$

$$\sin\theta = \langle yes | \varphi \rangle = \sqrt{\frac{m}{2^n}}$$

$$\cos\theta = \langle no | \varphi \rangle = \sqrt{1 - m/2^n}$$

$$V = 1 - 2|yes\rangle\langle yes|$$

$$W = 2|\varphi\rangle\langle\varphi| - 1$$

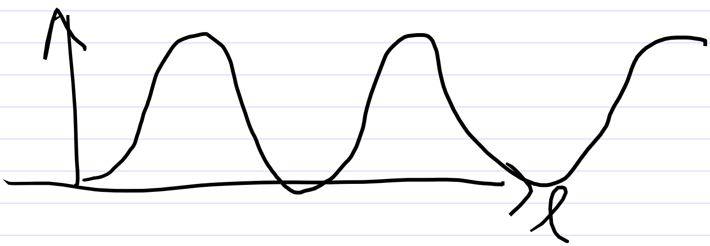


WV rotates by 2θ

$$(2l+1)\theta \simeq \pi/2$$

$$\theta \simeq \sqrt{m/2^n} = \sqrt{m/N}$$

$$l \simeq \pi/4 \sqrt{N/m}$$



$\langle yes|(WV)^{\ell}|\varphi\rangle$ is periodic

$\approx \sin 2\ell\theta$ in $\ell \pm \pi/\theta$

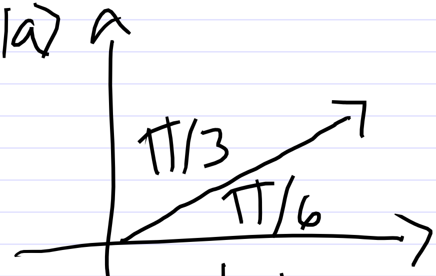
so period $= \dfrac{\pi 2^{n/2}}{\sqrt{m}}$.

Run QFT to get period, gives m

then Grover $\frac{\pi}{4}\sqrt{N/m}$ times

A special case: $m=1$ $n=2$ $N=4$

$\sin\theta = \langle a|\varphi\rangle = \frac{1}{2^{n/2}} = 1/2$ $\theta = \pi/6$



WV rotates by $\pi/3$

single WV exactly $|a\rangle$

Q.M.: $\underline{1}$ | classically expect: $\frac{1}{4}\cdot 1 + \frac{3}{4}\frac{1}{3}2 + \frac{1}{2}3 = 2\,1/4$

Just choose N/m states

$N = 200$ $\underline{m = 10}$ pick $\frac{200}{10} = 20$

prob of at least one $1 - \left(\frac{19}{20}\right)^{20} \approx 1 - e^{-1}$

prob of exactly one $\approx e^{-1}$ (Poisson)

Then run ordinary Grover

$\sqrt{N/m}$