$|\psi\rangle = \sum \gamma_x |x\rangle$ and $U_{FT}|\psi\rangle = \sum \tilde{\gamma}_x |x\rangle$

then $\tilde{\gamma}_x = \frac{1}{2^{n/2}} \sum_y e^{2\pi i x y / 2^n} \gamma_y$

$y = y_{n-1} \cdots y_0$

$y = \sum_k 2^k y_k$

$3218/10 = 321.8$

$10101/4 = 101.01$

$.1 = 2^{-1} = \frac{1}{2}$ $\quad .11 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$.101 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

$$U_{FT}|x\rangle = \frac{1}{2^{n/2}} \sum_{0 \le y < 2^n} e^{2\pi i xy/2^n} |y\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{y} e^{2\pi i x \sum_{k=0}^{n-1} y_k/2^{n-k}} |y_{n-1} \cdots y_0\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{\{y_j = 0\}}^{1} \prod_{k=0}^{n-1} e^{2\pi i x y_k/2^{n-k}} |y_{n-1} \cdots y_0\rangle$$

$$= \frac{1}{2^{n/2}} \bigotimes_{k} \left[\sum_{y_k = 0}^{1} e^{2\pi i x y_k/2^{n-k}} |y_k\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{k=0}^{n-1} \left(|0\rangle + e^{2\pi i x/2^{n-k}} |1\rangle \right)$$

[Later we'll use $x = x_{n-1} x_{n-2} \cdots x_1 x_0$,
and $e^{2\pi i\, 1101.1} = e^{\cancel{2\pi i\, 13} + 2\pi i \frac{1}{2}} = e^{2\pi i\, .1}$ (decimal)]

$$= \frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i x/2}\right)\left(|0\rangle + e^{2\pi i x/4}\right) \cdots \left(|0\rangle + e^{2\pi i x/2^n} |1\rangle\right)$$

$$U_{FT} |x_{n-1} \ldots x_0\rangle = \frac{1}{2^{n/2}} \sum_y e^{2\pi i x y/2^n} |y\rangle$$

$$= \frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i \cdot 0.x_0} |1\rangle\right)$$

$$\cdot \left(|0\rangle + e^{2\pi i \cdot 0.x_1 x_0} |1\rangle\right)$$

$$\vdots$$

$$\cdot \left(|0\rangle + e^{2\pi i \cdot 0.x_{n-1} \ldots x_0} |1\rangle\right)$$

each has two terms for $y_i = 0, 1$,

product generates $\sum_{0 \le y < 2^n}$

Circuit Diagram

$$H|x_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_0}|1\rangle\right) = \boxed{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_0}|1\rangle\right)}$$

define phase operator $V_k = \begin{pmatrix} 1 & \\ & e^{\pi i/2^k} \end{pmatrix}$

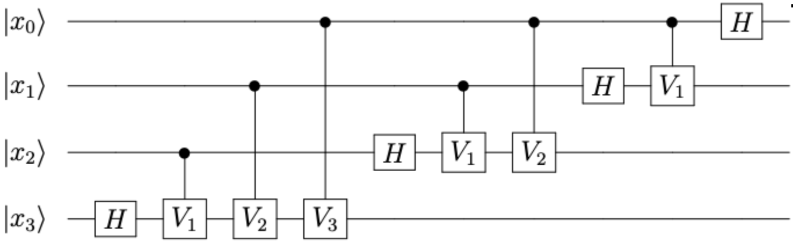
$$V_1^{x_0} H|x_1\rangle = V_1^{x_0}\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_1}|1\rangle\right)$$

$$\boxed{= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_1 x_0}|1\rangle\right)}$$

only adds $2\pi i/4 = 2\pi i \cdot .01$ if $x_0 = 1$

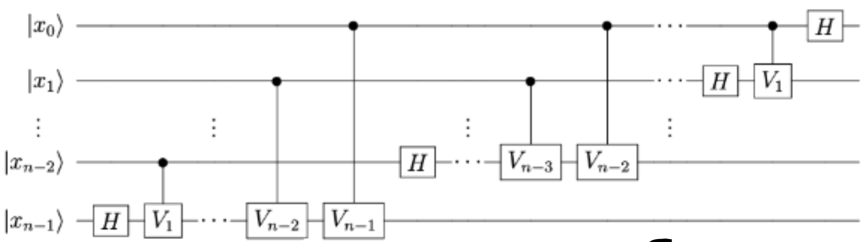
$$V_2^{x_0} V_1^{x_1} H|x_2\rangle = \boxed{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.x_2 x_1 x_0}|1\rangle\right)}$$

$$V_3^{x_0} V_2^{x_1} V_1^{x_2} H|x_3\rangle = \boxed{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.x_3 x_2 x_1 x_0}|1\rangle\right)}$$



$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_0}|1\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_1x_0}|1\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_2x_1x_0}|1\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.x_3x_2x_1x_0}|1\rangle\right)$$

$$= \frac{1}{2^2}\sum_{0\le y<2^4} e^{2\pi i xy/2^4}|y\rangle$$



$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_0}|1\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1x_0}|1\rangle\right)$$
$$\vdots$$
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_{n-2}\ldots x_0}|1\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_{n-1}\ldots x_0}|1\rangle\right)$$

$$U_{FT}|x_{n-1}\ \ldots\ x_0\rangle$$

$$\left[\begin{aligned} \#H &= n \\ \#V &= 0+1+\ldots+n-1 = \frac{n(n-1)}{2} \end{aligned}\right]$$

$$= \left(H|x_0\rangle\right)\left(V_1^{x_0}H|x_1\rangle\right)\left(V_2^{x_0}V_1^{x_1}H|x_2\rangle\right)$$
$$\cdots\left(V_{n-2}^{x_0}V_{n-3}^{x_1}\cdots V_1^{x_{n-3}}H|x_{n-2}\rangle\right)$$
$$\left(V_{n-1}^{x_0}\ldots V_1^{x_{n-2}}H|x_{n-1}\rangle\right)$$

For $n=4$ qubits, $\#H\text{'s} = 4$

$\#V\text{'s} = 6$

For n qubits, $\#H\text{'s} = n$

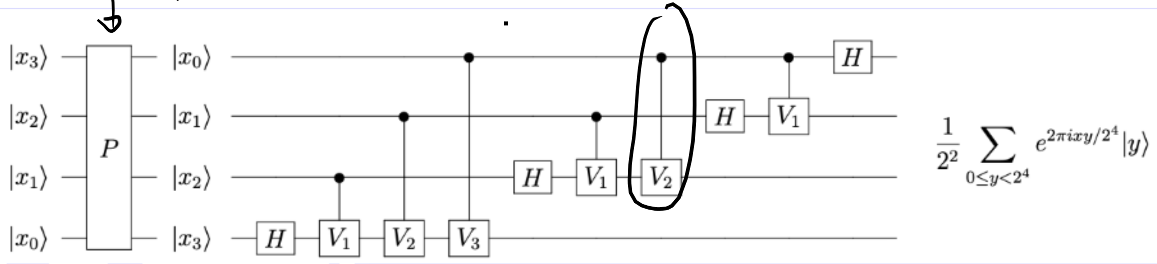
$$\#V\text{'s} = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

So Q.F.T. is quadratic in n

time $\propto$ #gates.

Further simplifications next time.

permutes



$O(n^2)$ QFT is not the hard part

U_f for $f(x) = b^x \bmod N$

is $O(n^3)$

#H + #V gates is $O(n^2)$,
= time it takes to implement U_{FT}.

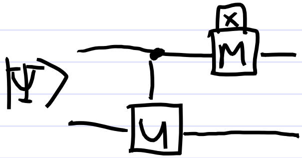
Simplification: Can eliminate all 2-qubit gates in favor of classically controlled 1-qubit gates.

Special case: qubit acts only as control bit of C^U gates before measurement.

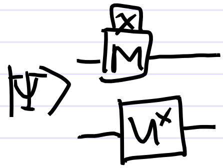
Then measuring first, and replace by classically controlled gates gives same states with same probability:

$$|\Psi\rangle_n = \alpha_0 |0\rangle |\Phi_0\rangle_{n-1} + \alpha_1 |1\rangle |\Phi_1\rangle_{n-1}$$

$$C^U |\Psi\rangle_n = \alpha_0 |0\rangle |\Phi_0\rangle_{n-1} + \alpha_1 |1\rangle U |\Phi_1\rangle_{n-1}$$



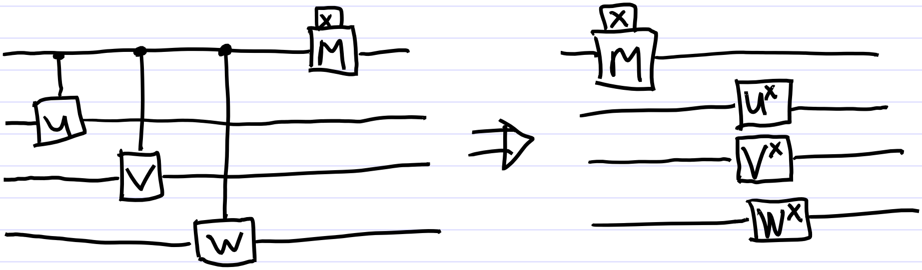
result	w/ prob
$\lvert 0\rangle \lvert \Phi_0\rangle$	$\lvert\alpha_0\rvert^2$
$\lvert 1\rangle U \lvert \Phi_1\rangle$	$\lvert\alpha_1\rvert^2$



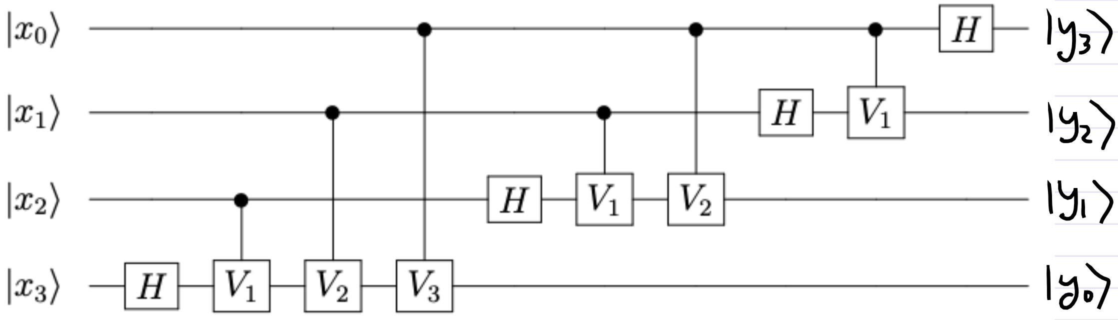
result	w/ prob
$\lvert 0\rangle \lvert \Phi_0\rangle$	$\lvert\alpha_0\rvert^2$
$\lvert 1\rangle U \lvert \Phi_1\rangle$	$\lvert\alpha_1\rvert^2$

same final state w/ same probability!

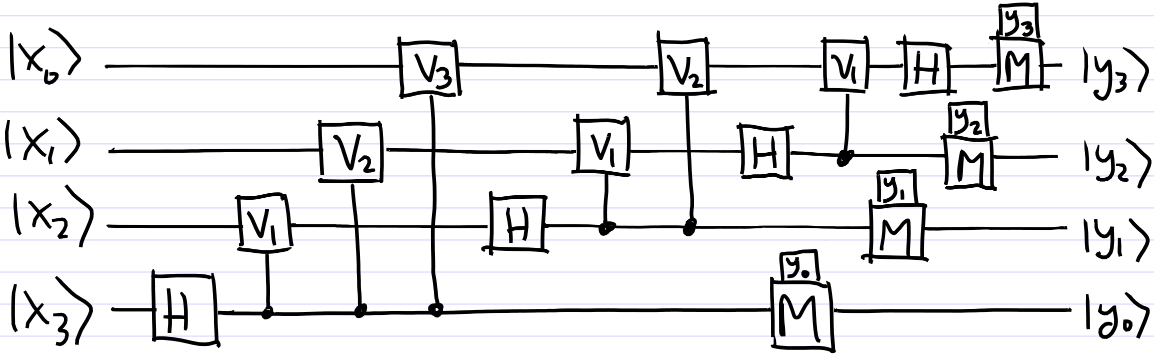
So can replace, e.g.,



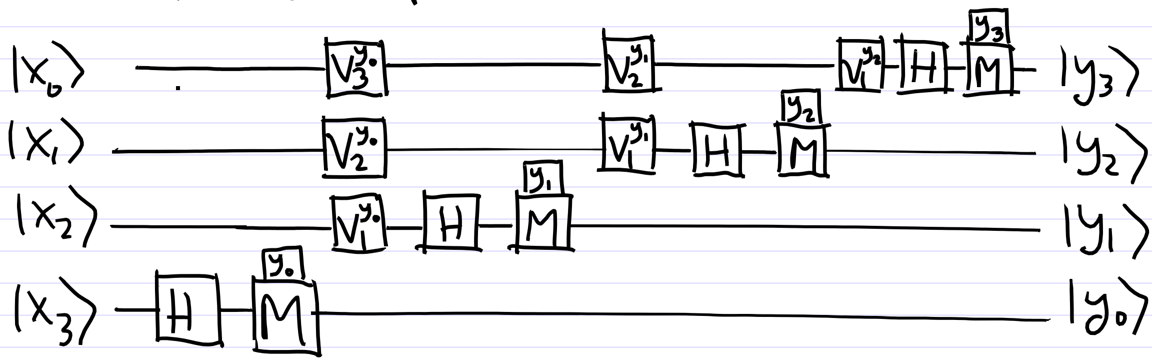
where U^x means apply U "by hand" only if $x = 1$



Controlled phases are symmetric:



Now swap them out:



Now only 1-qubit gates!

But wait.... there's more:

"Qubit Recycling"

$$|x_3\rangle - \boxed{H} - \overset{\boxed{y_0}}{\boxed{M}} - |y_0\rangle$$

After measurement, no longer used, so reinitialize and use again.

$$|x_2\rangle - \boxed{V_1^{y_0}} - \boxed{H} - \overset{\boxed{y_1}}{\boxed{M}} - |y_1\rangle$$

and again, ... n times to get all of $|y\rangle$.

See arXiv: 1111.4147 for experimental realization