Measurement in different bases

Measurement in Z basis:
Expand $|\psi\rangle$ in eigenstates $|0\rangle, |1\rangle$ of the Z operator, and measure $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

Measurement in X basis:
Expand $|\psi\rangle$ in eigenstates $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ of the X operator

$$|\psi\rangle = \alpha_+|+\rangle + \alpha_-|-\rangle \ .$$

Note that $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$, so that

$$H|\psi\rangle = \alpha_+|0\rangle + \alpha_-|1\rangle \ ,$$

and measurement of $H|\psi\rangle$ in the original-Z basis is governed by the amplitudes from the X-basis (i.e., $\alpha_\pm = \frac{1}{\sqrt{2}}(\alpha_0 \pm \alpha_1)$).

Measurement in Y basis:
Expand $|\psi\rangle$ in eigenstates $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ of the Y operator

$$|\psi\rangle = \alpha_{+i}|+i\rangle + \alpha_{-i}|-i\rangle \ .$$

Note that $SH|0\rangle = |+i\rangle$ and $SH|1\rangle = |-i\rangle$, so apply $(SH)^{-1} = HS^\dagger$ to write

$$HS^\dagger|\psi\rangle = \alpha_{+i}|0\rangle + \alpha_{-i}|1\rangle \ .$$

and measurement of $HS^\dagger|\psi\rangle$ in the original-Z basis is governed by the amplitudes from the Y-basis (i.e., $\alpha_{\pm i} = \frac{1}{\sqrt{2}}(\alpha_0 \pm i\alpha_1)$).

Measurement in an arbitrary basis:
Suppose $uZu^\dagger = \hat{n}\cdot\vec{\sigma}$, and denote the eigenstates of $\hat{n}\cdot\vec{\sigma}$ by $|\pm\hat{n}\rangle$, so that

$$|\psi\rangle = \alpha_{+\hat{n}}|+\hat{n}\rangle + \alpha_{-\hat{n}}|-\hat{n}\rangle \ .$$

Then‡ $u|0\rangle = |+\hat{n}\rangle$ and $u|1\rangle = |-\hat{n}\rangle$, so that

$$u^\dagger|\psi\rangle = \alpha_{+\hat{n}}|0\rangle + \alpha_{-\hat{n}}|1\rangle \ ,$$

and measurement of $u^\dagger|\psi\rangle$ in the original-Z basis is governed by the amplitudes $\alpha_{\pm\hat{n}}$ from the $\hat{n}\cdot\vec{\sigma}$-basis.

---

‡ because $\hat{n}\cdot\vec{\sigma}|\pm\hat{n}\rangle = uZu^\dagger u|0,1\rangle = uZ|0,1\rangle = \pm u|0,1\rangle = \pm|\pm\hat{n}\rangle$