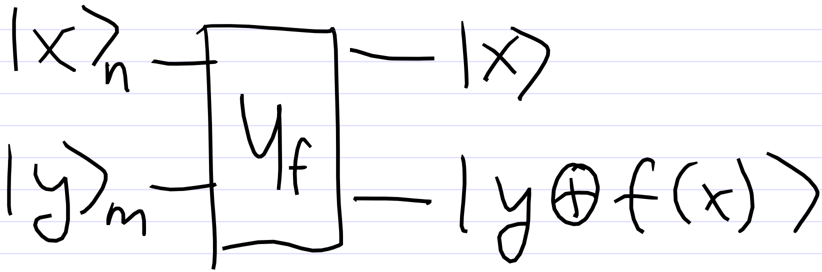


Deutsch (mid 1980's)

1 bit $\to$ 1 bit function

[e.g., y = millionth bit of $\sqrt{2+x}$]

$y = f(x)$ $\quad m = n = 1$

	$x=0$	$x=1$	
f_0	0	0	const $f(x)=0$
f_1	0	1	identity $f(x)=x$
f_2	1	0	Not $f(x)=1-x$
f_3	1	1	const $f(x)=1$

Classically need two invocations of f to distinguish (f_0, f_3) from (f_1, f_2)

QM: Can distinguish with just 1 invocation of f

[factor of 2 (?) speedup]

$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

$$U_{se}\ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$ to initialize output qubit

$$U_f|x\rangle|-\rangle = |x\rangle\frac{1}{\sqrt{2}}\left(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle\right)$$

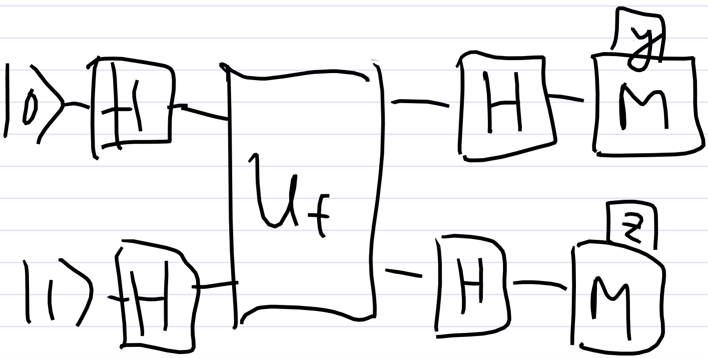
$$\begin{array}{l} f(x) = 0 \Rightarrow 1 \\ f(x) = 1 \Rightarrow -1 \end{array} = (-1)^{f(x)}|x\rangle|-\rangle$$

(phase kickback)

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}|-\rangle \xrightarrow[U_f]{} \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}|-\rangle$$

$$f(0) = f(1) \qquad (-1)^{f(0)}\frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{H} \pm|0\rangle$$

$$f(0) \neq f(1) \qquad (-1)^{f(0)}\frac{|0\rangle - |1\rangle}{\sqrt{2}} \xrightarrow{H} \pm|1\rangle$$



$y=0$ (f_0, f_3)

$y=1$ (f_1, f_2)

$z=1$ (no info)

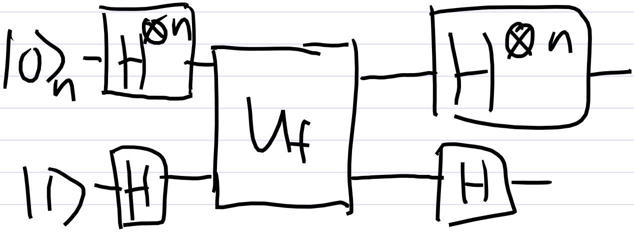
We've learned const vs non-const (even vs odd parity) in one invocation of U_f

1992 Deutch-Josza

input is n-bit
output is 1-bit
$f(x)$ is either constant
or "balanced"
balanced means f is 0 on half
the 2^n values and 1 on other half

Classically, takes $2^{n-1}+1$
invocations of f to distinguish
Constant from balanced

QM: just 1



Consider ${}_n\langle 0| H^{\otimes n} U_f H^{\otimes n} |0\rangle_n$

suppose f is const

$U_f H^{\otimes n} |0\rangle_n |-\rangle = (-1)^{f(0)} H^{\otimes n} |0\rangle_n |-\rangle$

so ${}_n\langle 0| H^{\otimes n} U_f H^{\otimes n} |0\rangle_n = \pm 1$

$\Rightarrow$ we measure 0 100% of time

---

but ${}_n\langle 0| H^{\otimes n} U_f H^{\otimes n} |0\rangle_n = 0$

when f is balanced, because equal # ± 1

3 qubit example

$$H^{\otimes 3}|0\rangle_3 = \frac{1}{\sqrt{8}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big)$$

Suppose f is balanced:

$$U_f H^{\otimes 3}|0\rangle_3 = \frac{1}{\sqrt{8}}\Big(-|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle\Big)$$

$$\langle 0| H^{\otimes 3} U_f H^{\otimes 3} |0\rangle$$
$$= \frac{1}{8}\left(-1 + 1 + 1 - 1 + 1 - 1 - 1 + 1\right)$$
$$= 0$$

So measure the input qubits,
if all 0 it's constant
otherwise it's balanced.

'93 Bernstein-Vazarani

again n bit $\rightarrow$ 1 bit

$$f(x) = a \cdot x = \bigoplus_i a_i x_i$$

Question: what is a?

$x = (0 \ldots 0 1) \Rightarrow a_0$

$x = (0 \ldots \underset{m^{th}}{1} \ldots) \Rightarrow a_m$

Classicaly: n calls to f

QM: just 1 call to f

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

write as $$H|x\rangle = \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \sum_{y=0}^{1} (-1)^{xy} |y\rangle$$

$$x = x_{n-1} \cdots x_1 x_0$$

$$H^{\otimes n} |x\rangle_n = \frac{1}{2^{n/2}} \sum_{y_{n-1}=0}^{1} (-1)^{x_{n-1} y_{n-1}} |y_n\rangle \cdots \sum_{y_0=0}^{1} (-1)^{x_0 y_0} |y_0\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{\{y_j=0\}}^{1} (-1)^{x_{n-1} y_{n-1} + \ldots x_0 y_0} |y_{n-1}\rangle \cdots |y_0\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{y=0}^{2^n - 1} (-1)^{x \cdot y} |y\rangle$$

$$H^{\otimes n}|x\rangle = \frac{1}{2^{n/2}} \sum_y (-1)^{x\cdot y} |y\rangle$$

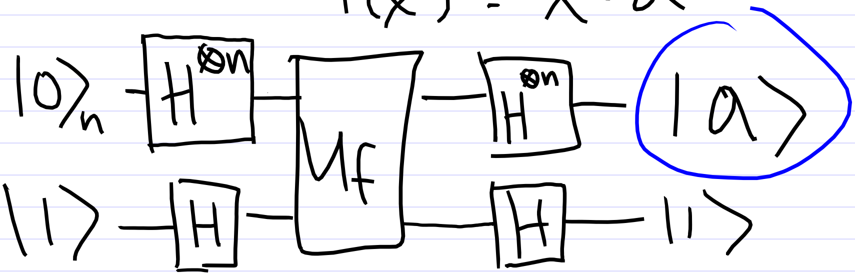
$$\underbrace{H^{\otimes n} \frac{1}{2^{n/2}} \sum_y (-1)^{x\cdot y} |y\rangle}$$

$$= H^{\otimes n} H^{\otimes n} |x\rangle = |x\rangle$$

$x \to a \quad y \to x$

$$H^{\otimes n} \frac{1}{2^{n/2}} \sum_x (-1)^{a\cdot x} |x\rangle = |a\rangle$$

$$f(x) = x \cdot a$$



$$U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle = (-1)^{x \cdot a}|x\rangle|-\rangle$$

$$U_f H^{\otimes n}|0\rangle|-\rangle = \frac{1}{2^{n/2}}\sum_x (-1)^{a \cdot x}|x\rangle|-\rangle$$

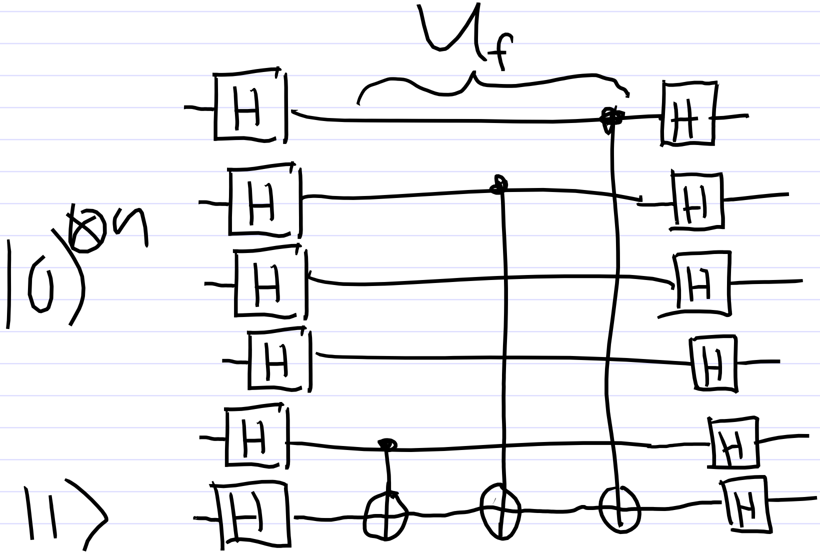
$$H^{\otimes n}\frac{1}{2^{n/2}}\sum_x (-1)^{a \cdot x}|x\rangle$$

$$= |a\rangle$$ so measure input qubits

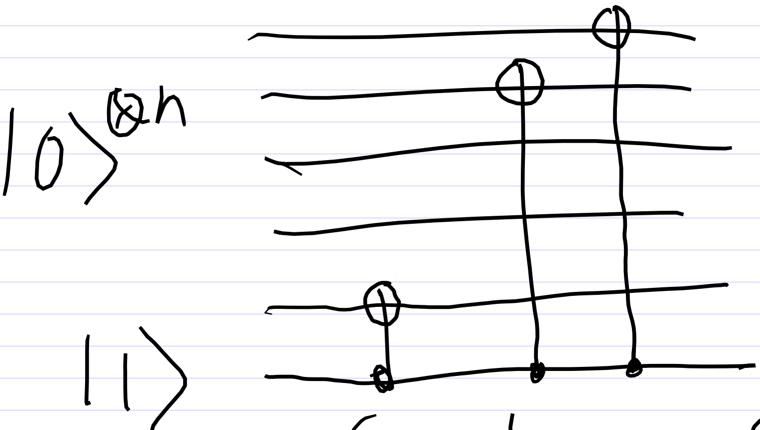
after just one invocation of U_f

(Alternatively)

$a = (11001) \quad f(x) = x \cdot a$



$=$



$|a\rangle$

factor of n

"Brute force" derivation of

$$H^{\otimes n} \frac{1}{2^{n/2}} \sum_y (-1)^{x \cdot y} |y\rangle = |x\rangle$$

$$H^{\otimes n} \frac{1}{2^{n/2}} \sum_y (-1)^{x \cdot y} |y\rangle$$

$$= \frac{1}{2^n} \sum_y \sum_z (-1)^{x \cdot y} (-1)^{y \cdot z} |z\rangle$$

$$\left[\text{but } \frac{1}{2} \sum_{y_i} (-1)^{y_i(x_i - z_i)} = \begin{matrix} \frac{1}{2}(1+1) & x_i = z_i \\ \frac{1}{2}(1-1) & x_i \neq z_i \end{matrix} = \delta_{x_i z_i} \right]$$

so

$$= \sum_z \delta_{xz} |z\rangle = |x\rangle$$