n qubit error-correcting code

n = # qubits, low error rate, so only single qubit errors:

$$|\psi\rangle \to \left(\mathbb{1} + \sum_{j=0}^{n-1} \left(\varepsilon_x^j X_j + \varepsilon_y^j Y_j + \varepsilon_z^j Z_j\right)\right)|\psi\rangle$$

(Any of single X, Y, Z error on any of n-qubits in codeword)

To have orthogonal subspaces to correct any of 3n+1 errors, need

$$2(3n+1) \leq 2^n$$

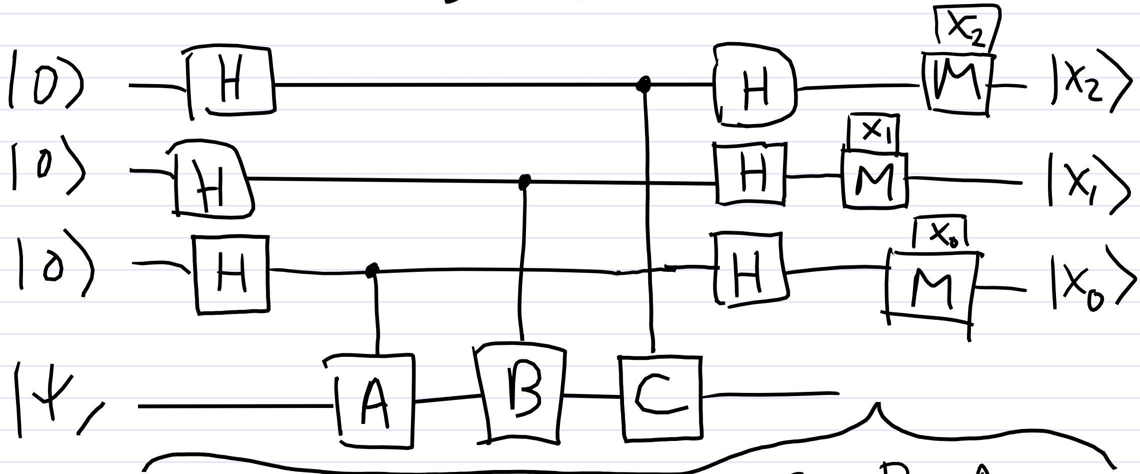
($3n$ ← X_i, Y_i, Z_i)

i.e., $n \geq 5$

and there will be a minimal n = 5 code that corrects all three types of error.

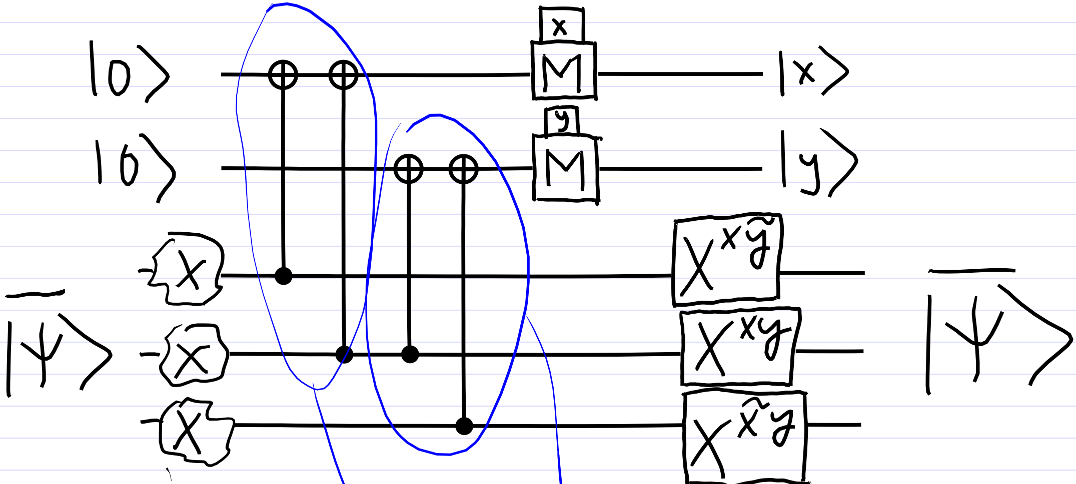
Multiple operator measurements

$$A^2 = B^2 = C^2 = 1$$

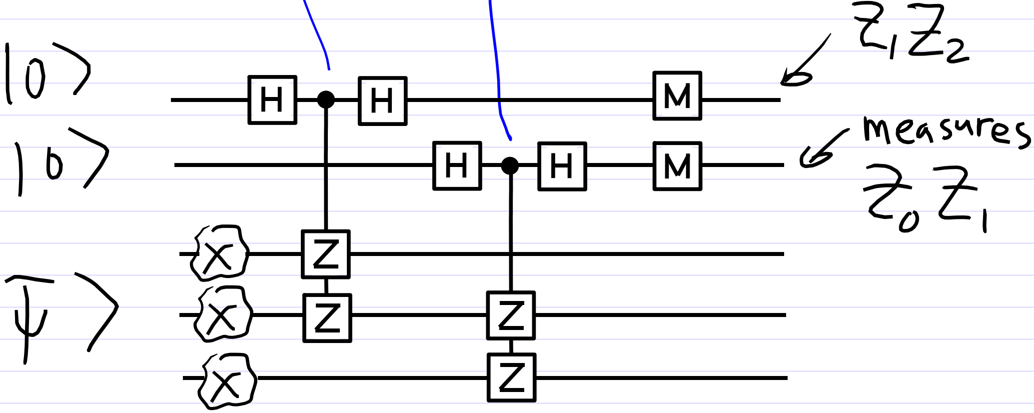


$$\sum_{x_2,x_1,x_0} |x_2\rangle|x_1\rangle|x_0\rangle\, P^C_{x_2} P^B_{x_1} P^A_{x_0} |\psi\rangle$$

⇒ If A, B, C are mutually commuting, then measuring x_0, x_1, x_2 projects onto their joint ±1 eigenspaces.


|0⟩
|0⟩
x
M
y
M
|x⟩
|y⟩
X
X
X
$|\bar{\psi}\rangle$
$X^{x\bar{y}}$
X^{xy}
$X^{\bar{x}y}$
$|\bar{\psi}\rangle$

Use $X = HZH$


|0⟩
|0⟩
H
H
H
H
M
M
Z
Z
Z
Z
X
X
X
$|\bar{\psi}\rangle$
"measures"
$Z_1 Z_2$
measures
$Z_0 Z_1$

$$V\left(\frac{1+(-1)^x V}{2}\right) = (-1)^x\left(\frac{1+(-1)^x V}{2}\right)$$

$$P_x^V$$

Specify 5 qubit code

$$M_0 = Z_1 X_2 X_3 Z_4 \qquad \underline{M_i^2 = 1}$$

$$M_1 = Z_2 X_3 X_4 Z_0$$

$$M_2 = Z_3 X_4 X_0 Z_1$$

$$M_3 = Z_4 X_0 X_1 Z_2$$

$$[X_i, Z_j] = 0 \quad i \neq j$$

Commutator

$$[A,B] = AB - BA$$

$$[M_i, M_j] = 0$$

$M_4 = Z_0 X_1 X_2 Z_3$?

not independent $= M_0 M_1 M_2 M_3$

Code words

$$|\bar{0}\rangle = \frac{1}{4}(1+M_0)(1+M_1)(1+M_2)(1+M_3)|00000\rangle$$

$$|\bar{1}\rangle = \frac{1}{4}(1+M_0)(1+M_1)(1+M_2)(1+M_3)|11111\rangle$$

Normalized? $(1+M_i)^2 = 2(1+M_i)$

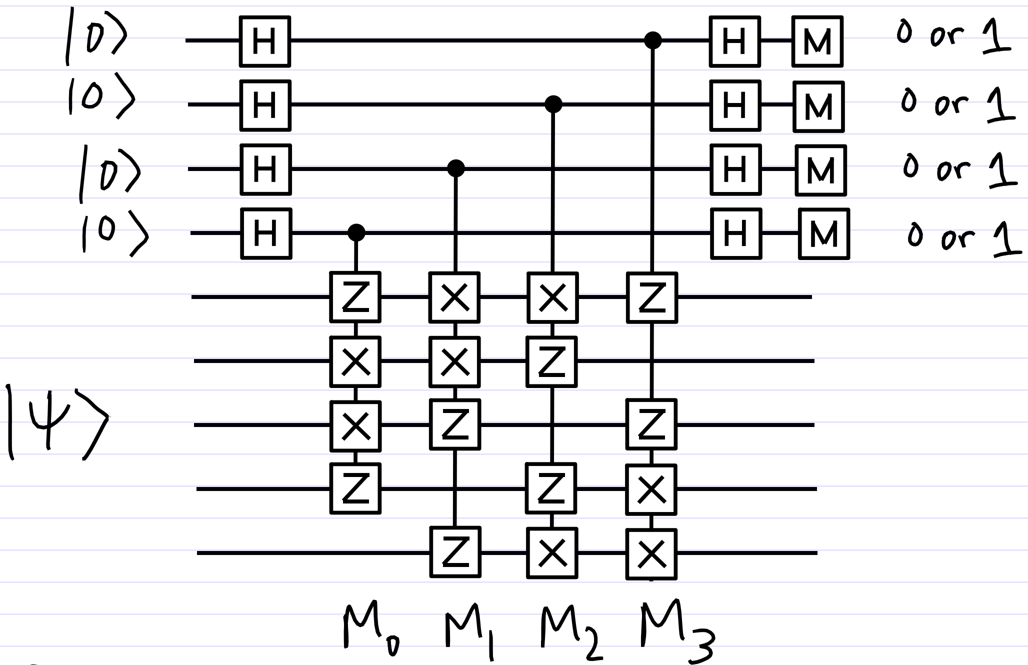
$$\langle\bar{0}|\bar{0}\rangle = \frac{1}{16} 16 \langle 0^5|\prod_i (1+M_i)|0^5\rangle = \underline{1}$$

$$\langle\bar{1}|\bar{1}\rangle = \underline{1} \qquad \bar{X} = X_0 X_1 X_2 X_3 X_4$$

$$\langle\bar{1}|\bar{0}\rangle = \langle\bar{0}|\bar{1}\rangle = 0 \qquad \bar{Z} = Z_0 Z_1 Z_2 Z_3 Z_4$$

$$|\bar{1}\rangle = \bar{X}|\bar{0}\rangle \qquad \bar{Z}|\bar{0}\rangle = |\bar{0}\rangle \qquad [\bar{X}, M_i] = 0$$

$$|\bar{0}\rangle = \bar{X}|\bar{1}\rangle \qquad \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle \qquad [\bar{Z}, M_i] = 0$$



5-qubit code, encoding circuit.

To initialize state to $|\bar{0}\rangle$:

measure M_i's, projects to $\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$ space.

measure $\bar{Z}$, gives $|\bar{0}\rangle$ or $|\bar{1}\rangle$

if $|\bar{1}\rangle$, apply $\bar{X}|\bar{1}\rangle = |\bar{0}\rangle$

Next: We'll show that any of the fifteen

$X_i, Y_i, Z_i \quad i = 0, \ldots, 4$

has a unique signature in eigenvalues of the M_i.

e.g. $|\psi\rangle \to X_3 |\psi\rangle$, then since X_3 commutes with M_0, M_1, M_3, and anti-commutes with M_2, only M_2 flips sign, and the M_i would be measured as

$$(+, +, -, +)$$

$M_0 = Z_1 X_2 X_3 Z_4$ $\quad$ $M_2 = Z_3 X_4 X_0 Z_1$

$M_1 = Z_2 X_3 X_4 Z_0$ $\quad$ $M_3 = Z_4 X_0 X_1 Z_2$

Now see that the M_i characterize the 16 spaces $\mathbb{1}$ X_i Y_i Z_i :

($\mathbb{1}$ → uncorrupted; X_i Y_i Z_i → 15 corruptions)

	X_0	Y_0	Z_0	X_1	Y_1	Z_1	X_2	Y_2	Z_2	X_3	Y_3	Z_3	X_4	Y_4	Z_4	$\mathbb{1}$
M_0	+	+	+	−	−	+	+	−	−	+	−	−	−	−	+	+
M_1	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+
M_2	+	−	−	−	−	+	+	+	+	−	−	+	+	−	−	+
M_3	+	−	−	+	−	−	−	−	+	+	+	+	−	−	+	+

each column is a unique error signature. Just look at whether the given operator commutes or anti-commutes with M_i.

e.g., (start of 1st column)

$$M_0 X_0 |\psi\rangle = X_0 M_0 |\psi\rangle = +X_0|\psi\rangle$$

$$M_1 X_0 |\psi\rangle = -X_0 M_1 |\psi\rangle = -X_0|\psi\rangle$$

Recall $|\bar{0}\rangle = \frac{1}{4}\prod_i (1+M_i)|0^5\rangle$

$$|\bar{1}\rangle = \frac{1}{4}\prod_i (1+M_i)|1^5\rangle$$

have $M_0, M_1, M_2, M_3 = +1, +1, +1, +1$

Suppose: measure M_0, M_1, M_2, M_3
as $+1, -1, +1, -1$

How to correct error?

Well $+-+-$ is the X_2 column, so the state has an X_2 error

$$X_2|\psi\rangle$$

To correct, apply X_2

$$X_2 X_2|\psi\rangle = |\psi\rangle$$

How to implement arbitrary U?

- Universal gate set

  CNOT, Z, H, T $T^4 = Z$ ($\pi/8$ gate)

  gives any U

difficulty with 5-qubit code:

"hard" to get H, CNOT

instead use:

7-qubit "Steane code"