

Appendix I (linear algebra)

$$|\psi\rangle = \sum_{x=0}^{\infty} \alpha_x |x\rangle$$

$$|\varphi\rangle = \sum_{x=0}^{\infty} \beta_x |x\rangle$$

$\alpha|\psi\rangle + \beta|\varphi\rangle$
is also a vector

inner product $\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$
 $\langle\varphi|\varphi\rangle > 0$ if $|\varphi\rangle \neq 0$

"Anti-linear": $\langle\varphi|\alpha\psi_1 + \beta\psi_2\rangle = \alpha\langle\varphi|\psi_1\rangle + \beta\langle\varphi|\psi_2\rangle$

In components $\langle\varphi|\psi\rangle = \sum_x \beta_x^* \alpha_x$

$$\langle\varphi|\varphi\rangle = \sum_x |\alpha_x|^2$$

A linear transformation satisfies

$$A(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha A|\psi\rangle + \beta A|\varphi\rangle$$

If it preserves the norm of vectors

$\langle A\psi|A\psi\rangle = \langle\psi|\psi\rangle$, then called "Unitary"
(takes unit vectors to unit vectors)

Unitary U also preserves arbitrary inner products

$$\langle U\psi | U\psi \rangle = \langle \psi | \psi \rangle \\ = \langle \psi | U^\dagger U | \psi \rangle \quad \text{so } U^\dagger U = 1 = U U^\dagger$$

and $U^\dagger = U^{-1}$ where $U^\dagger = (U^T)^*$

ie. $(U^\dagger)_{ij} = U_{ji}^*$ $U_{ij} = \langle i | U | j \rangle$

Consider components in some basis

$$\alpha_i = \langle i | \psi \rangle, \quad \beta_j = \langle j | \psi \rangle$$

See how this works:

$$\sum_i \beta_i^* \alpha_i = \sum_{i,j,k} (U_{ij} \beta_j)^* U_{ik} \alpha_k = \sum_{j,k} \beta_j^* \underbrace{U_{ij}^* U_{ik}}_{\text{has to be } \delta_{jk}} \alpha_k$$

$$\text{So } \sum_i U_{ij}^* U_{ik} = \delta_{jk}$$

$$= \sum_i (U^\dagger)_{ji} U_{ik} \quad \text{or } U^\dagger U = 1$$

Slightly different from Hermitian, $H^\dagger = H$,
but a matrix can be both unitary
and Hermitian, e.g., Hadamard, Z , X

Appendix II

consider 2×2 unitary matrices, the group $U(2)$

$$u = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$$

$$u^\dagger u = \mathbb{1} \Rightarrow u = e^{i\varphi/2} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix}$$

8 real params,
4 conditions

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\det u = e^{i\varphi}$$

$$\varphi = 0 \Rightarrow SU(2)$$

write

$$u = u_0 \mathbb{1} + i \vec{v} \cdot \vec{\sigma}$$

$$= \begin{pmatrix} u_0 + i v_3 & i v_1 + v_2 \\ i v_1 - v_2 & u_0 - i v_3 \end{pmatrix}$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$= \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

$\det u = u_0^2 + \vec{v}^2 = 1$ (3-sphere S^3 embedded in \mathbb{R}^4)

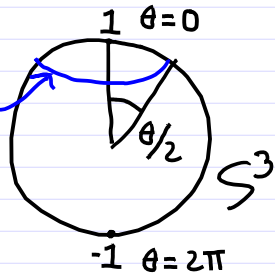
$$\begin{aligned} \text{check } u u^\dagger &= (u_0 \mathbb{1} + i \vec{v} \cdot \vec{\sigma})(u_0 \mathbb{1} - i \vec{v} \cdot \vec{\sigma}) \\ &= (u_0^2 + v^2) \mathbb{1} = \mathbb{1} \end{aligned}$$

use polar coordinates

$$u_0 = \cos \frac{\theta}{2}$$

$$\vec{v} = \sin \frac{\theta}{2} \hat{n}$$

unit vector
in 3d
on S^2



so finally

$$\begin{aligned}
 u &= u_0 \mathbb{1} + i \vec{v} \cdot \vec{\sigma} \\
 &= \cos \frac{\theta}{2} \mathbb{1} + i \sin \frac{\theta}{2} \hat{n} \cdot \vec{\sigma} \\
 &= e^{i(\hat{n} \cdot \vec{\sigma}) \frac{\theta}{2}}
 \end{aligned}$$

(where from piazza post:)

[PG] Note that a matrix exponential is defined in terms of its power series expansion, so in general $\exp i(\hat{n} \cdot \vec{\sigma})x = I + i(\hat{n} \cdot \vec{\sigma})x + (i(\hat{n} \cdot \vec{\sigma})x)^2/2! + (i(\hat{n} \cdot \vec{\sigma})x)^3/3! + (i(\hat{n} \cdot \vec{\sigma})x)^4/4! + \dots$ (where \hat{n} is a unit vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$).

Using $\sigma_i \sigma_j = I \delta_{ij} + i \epsilon_{ijk} \sigma_k$, we see that $(\hat{n} \cdot \vec{\sigma})^2 = n_i n_j I \delta_{ij} = (\hat{n} \cdot \hat{n})I = I$, and the above becomes

$$\exp i(\hat{n} \cdot \vec{\sigma})x = I + i(\hat{n} \cdot \vec{\sigma})x - Ix^2/2! - i(\hat{n} \cdot \vec{\sigma})x^3/3! + Ix^4/4! + \dots$$

$$= I(1 - x^2/2! + x^4/4! - \dots) + i\hat{n} \cdot \vec{\sigma}(x - x^3/3! + \dots)$$

$$= I \cos(x) + i\hat{n} \cdot \vec{\sigma} \sin(x)$$

i.e., since $(\hat{n} \cdot \vec{\sigma})^2 = \mathbb{1}$, $i\hat{n} \cdot \vec{\sigma}$ plays the role of i

generalizing $e^{ix} = \cos x + i \sin x$

where

$$\begin{aligned}
 e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\
 &= \sum_{n=0}^{\infty} \frac{x^n}{n!}
 \end{aligned}$$

Appendix III

$\sigma_x, \sigma_y, \sigma_z$ basis for traceless
2x2 Hermitian matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{tr} A = \sum_k A_{kk}$$

satisfies $\text{tr} AB = \text{tr} BA$

$$\sigma_x \sigma_y = i\sigma_z, \sigma_y \sigma_z = i\sigma_x, \sigma_z \sigma_x = i\sigma_y; \sigma_i^2 = \mathbb{1}$$

more compactly:

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k$$

$$\text{tr} \sigma_i \sigma_j = 2\delta_{ij}$$

$$\epsilon_{xyz} = 1 = -\epsilon_{yxz} = \dots$$

$$\text{Let } \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\vec{a} \cdot \vec{\sigma} = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z$$

$$= \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}$$

Now from

$$\begin{aligned}(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) &= a_i b_j \sigma_i \sigma_j \\ &= a_i b_j (\delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k) \\ &= \vec{a} \cdot \vec{b} \mathbb{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}\end{aligned}$$

(where $(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k$)

Since the σ_i are traceless,

$$\text{tr}[(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma})] = \vec{a} \cdot \vec{b} \text{tr} \mathbb{1} = 2 \vec{a} \cdot \vec{b}$$

Now consider action of unitary U

$$U(\vec{a} \cdot \vec{\sigma})U^\dagger = \vec{a}' \cdot \vec{\sigma}$$

$$U(\vec{b} \cdot \vec{\sigma})U^\dagger = \vec{b}' \cdot \vec{\sigma}$$

$$\text{tr} UAU^\dagger = \text{tr} U^\dagger UA = \text{tr} A$$

(still Traceless Hermitian)

$$(UAU^\dagger)^\dagger = (U^\dagger)^\dagger A^\dagger U^\dagger = UAU^\dagger$$

$$\text{but } \text{tr}[U(\vec{a} \cdot \vec{\sigma})U^\dagger U(\vec{b} \cdot \vec{\sigma})U^\dagger] = \vec{a}' \cdot \vec{b}'$$

(Cyclicity of trace
and $U^\dagger U = 1$)

$$= \text{tr}(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b}$$

so $\vec{a} \cdot \vec{b} = \vec{a}' \cdot \vec{b}'$ preserves dot products
and the similarity transformation

$$U(\vec{a} \cdot \vec{\sigma})U^\dagger = \vec{a}' \cdot \vec{\sigma}$$

$$U(\vec{b} \cdot \vec{\sigma})U^\dagger = \vec{b}' \cdot \vec{\sigma}$$

induces a rotation $\vec{a}' = R_u \vec{a}$

$$\vec{b}' = R_u \vec{b}$$

of the 3D vectors \vec{a}, \vec{b}

Note that $\pm U$ induce the same
rotation so $SO(3) \approx SU(2)/\mathbb{Z}_2$

What rotation is it?

Let $u = e^{i(\hat{n} \cdot \sigma) \frac{\theta}{2}}$ take $\vec{a} = \hat{n}$.

Then $u(\hat{n} \cdot \vec{\sigma})u^\dagger = \hat{n} \cdot \vec{\sigma}$

so must be a rotation about the \hat{n} -axis.

For simplicity, take $\hat{n} = \hat{z}$, so $\hat{n} \cdot \vec{\sigma} = \sigma_z$,
and consider action on $\vec{a} = \hat{x}$, so $\vec{a} \cdot \vec{\sigma} = \sigma_x$

$$e^{i\frac{\theta}{2}\sigma_z} \sigma_x e^{-i\frac{\theta}{2}\sigma_z}$$

$$= \left(\cos\frac{\theta}{2} \mathbb{1} + i\sigma_z \sin\frac{\theta}{2} \right) \sigma_x \left(\cos\frac{\theta}{2} \mathbb{1} - i\sigma_z \sin\frac{\theta}{2} \right)$$

$$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \right) \sigma_x - 2\cos\frac{\theta}{2} \sin\frac{\theta}{2} \sigma_y$$

$$= \cos\theta \sigma_x - \sin\theta \sigma_y$$

$$\vec{a} = (1, 0, 0)$$

$$\vec{a}' = (\cos\theta, -\sin\theta, 0)$$

so $u(\hat{z}, \theta) = \exp(i\sigma_z \theta/2)$ induces a clockwise rotation by θ about the \hat{z} -axis

and $u(\hat{n}, \theta)$ a rotation by θ about the \hat{n} -axis

Can calculate $u(\hat{n}_2, \theta_2) u(\hat{n}_1, \theta_1) = u(\hat{n}_3, \theta_3)$

complete basis $|i\rangle$

$$\sum_i |i\rangle\langle i| = \mathbb{1}$$

$$|\psi\rangle = \sum_i \alpha_i |i\rangle \quad \left(\sum_j |j\rangle\langle j|\psi\rangle = \sum_j |j\rangle \sum_i \alpha_i \langle j|i\rangle = \sum_i \alpha_i |i\rangle \right)$$

$$u|i\rangle = \sum_j |j\rangle\langle j|u|i\rangle = \sum_j u_{ji} |j\rangle$$

$$u_{ji} \equiv \langle j|u|i\rangle$$

$$\text{so } u \sum_i \alpha_i |i\rangle = \sum_i \alpha_i u_{ji} |j\rangle = \sum_j u_{ij} \alpha_j |i\rangle$$

$$\text{and } \alpha_i \rightarrow \sum_j u_{ij} \alpha_j \quad \Sigma = \begin{pmatrix} s_1 & & & \\ & \ddots & & \\ & & s_{m-1} & \\ & & & 0 \end{pmatrix}$$

$$M = U \Sigma V^T$$

if M is $m \times n$, U is $m \times m$

Proof: U, V defined by

$$M M^T = U \Lambda U^T$$

$$M^T M = V \Lambda V^T$$

$$s_i = \sqrt{\lambda_i} \quad \Sigma \approx \sqrt{\Lambda}$$

V is $n \times n$

Σ is $m \times n$

$$M = U \Sigma V^T$$

$$|\psi\rangle = M_{ij} |i\rangle |j\rangle = \sum_k s_k U_{ik} V_{jk}^* |i\rangle |j\rangle = s_k |u_k\rangle |v_k^*\rangle$$

$$\text{where } |u_k\rangle \equiv \sum_i U_{ik} |i\rangle \quad |v_k^*\rangle \equiv \sum_j V_{jk}^* |j\rangle$$

product state iff single $s_k \neq 0$

$$|\psi\rangle = \alpha_{ij} |i\rangle |j\rangle$$

$$2 \text{ qubit } \alpha = u \Sigma v^T \quad \alpha_{ij} = \sum_k U_{ik} s_k V_{jk}^*$$

$$\text{recall } u|k\rangle = |i\rangle \langle i|u|k\rangle = U_{ik} |i\rangle$$

$$|\psi\rangle = \sum_{ijk} U_{ik} s_k V_{jk}^* |i\rangle |j\rangle = \sum_k (u \otimes v^*) s_k |k\rangle |k\rangle$$

$$= (u \otimes v^*) (s_0 |0\rangle |0\rangle + s_1 |1\rangle |1\rangle)$$

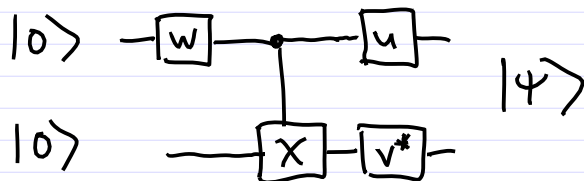
$$= (u \otimes v^*) C_{10} (s_0 |0\rangle + s_1 |1\rangle) |0\rangle$$

$$= (u \otimes v^*) C_{10} w_1 |0\rangle |0\rangle$$

$$\text{where } s_0 |0\rangle + s_1 |1\rangle = w_1 |0\rangle$$

all entanglement from CNOT

$$\uparrow w_1 \otimes \mathbb{1}$$



gives most general 2-qubit state