Physics 4481-7681; CS 4812 Problem Set 7

Four problems plus one bonus, covers lectures 24-26, due Wed evening 8 Dec 2021 at 23:00
[The problems are written as tutorials, so much shorter than they look.]

Problem 1: Jupyter notebook

Problem 2: Surface code I

At the 29:28–29:43 section in the video* linked from Lecture 23 (J. Martinis talk at Google LA, Oct '13), it is claimed that “if you can be at a .1% error here, and make a few thousand qubits, you can hold a qubit state, this fragile quantum state, for the lifetime of the universe.” Use the hints below to verify a ballpark estimate of the time that a logical qubit state can persist without decohering in their framework.

[Comments: The time estimate appears to be based on having physical qubits that remain coherent for 10µs with 99.9% probability. That means that the surface code error correction cycle can be applied every 10µs, and if in each such cycle there’s a $P_L = 10^{-5}$ chance of a logical error, then the state can persist for order $10^5$ cycles $\times 10\mu s = 1$ sec. A $P_L = 10^{-10}$ probability of logical error would give $10^5$ sec, of order a day (86400s). The slides in the 29:00 range are from their review article, arXiv:1208.0928, as used in class. But there are a few problems with the slide at the 29:07 point, e.g., it incorrectly has 15 orders of magnitude between 1 sec and the age of the universe ($\sim 10^{10}$ years), and in addition the green dot showing the claimed result is somewhat below even that incorrect line.†]

A rough estimate of the logical error rate $P_L$ can be made as follows (using just classical probability). As explained in class, a large probability of logical error means roughly half of the qubits along any row of the $d \times d$ array have errors. If the failure probability of a single qubit is some small $p$, then there’s a $p_e = 8p$ probability of a failure over any of the eight parts of the measurement cycle (four $Z$ and four $X$ measurements). The overall probability of logical failure is then $p_L = 2d(d/2)p^d_e/2$, since there are $(d/2)$ ways for $d/2$ qubits in a row of $d$ total to have errors, and there are $d$ rows and $d$ columns.

Using this crude estimate, determine $d$ (the linear size of the data qubit array) and $n_q = (2d - 1)^2$ (the total number of data and measurement qubits in the square array) for logical error rates of $10^{-10}$, $10^{-20}$, $10^{-23}$ for a single qubit error rate of $p = .001$ as above.

With this assumed value of $p$ for 10µs coherence time, how many qubits $n_q$ would it require for the surface code logical qubit to last of order lifetime of the universe?‡

---

* https://www.youtube.com/watch?v=HQmFEt6l6Tw
† But the final result is consistent, so they must have done the correct calculation at some point (and agrees with the more refined error analysis later reported in arXiv:1401.2466 : “Quantifying the effects of local many-qubit errors and non-local two-qubit errors on the surface code”, by A. Fowler and J. Martinis), despite the errors on slide (the black line for the universe should have been labelled $10^{23}$, and it should have gone through the green disk; red and blue lines presumably also off).
‡ You can also compare with the later more refined error analysis in arXiv:1401.2466
Problem 3: Surface Code II

As discussed in class (lecture 26), in section XIVb of arXiv:1208.0928 (surface code review), the authors consider the transformation of the 2-qubit operators $\sigma_a \otimes \sigma_b$ (where $\sigma = \{1, X, Y, Z\}$) under a “braiding” operation in which a Z-cut qubit is taken around an X-cut qubit. (The “cuts” indicate a 4-Z and 4-X measurement qubits turned off, to permit a larger logical Hilbert space.) They show that the effect of this operation is to transform the following 2-qubit operators as

$$X \otimes 1 \rightarrow X \otimes X, \quad 1 \otimes X \rightarrow 1 \otimes X, \quad 1 \otimes Z \rightarrow Z \otimes Z, \quad Z \otimes 1 \rightarrow Z \otimes 1.$$ 

It is worth looking at figures 21–23 of the above reference to see how these relations are determined, but that is not the objective of this exercise. Instead the (much simpler) objective is to verify that the above four relations fully characterize a cNOT gate $C$. As they say, “One way to test an experimental cNOT gate is to perform the cNOT on each of the four two-qubit basis states, and then do projective measurements of the result onto each of the four basis states. These sixteen experiments can be compared to the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

to verify that the cNOT has been implemented correctly.”

An equivalent method\(^\dagger\) is to examine the transformation of the 2-qubit operators under the action of $C$. The results above indicate that

$$C^\dagger(X \otimes 1)C = X \otimes X, \quad C^\dagger(1 \otimes X)C = 1 \otimes X, \quad C^\dagger(1 \otimes Z)C = Z \otimes Z, \quad C^\dagger(Z \otimes 1)C = Z \otimes 1.$$ 

By taking products, and using standard operator relations, show that the above four transformations are sufficient to imply the correct relations for all sixteen 2-qubit operators $C^\dagger(\sigma_a \otimes \sigma_b)C = \sigma'_a \otimes \sigma'_b$.

\(^\dagger\) corresponding to the Heisenberg picture, whereas the above action on states corresponds to the Schrödinger picture: if all the operators transform correctly, that means they act as though $C$ has been applied to the states.
**Problem 4: Surface code III**

In class (lec26), we continued with the surface code, in particular how to determine the number of logical qubits, and how to construct logical $\mathbf{X}$ and $\mathbf{Z}$ operators. The notes also cover the article arXiv:1912.09410 (linked from course webpage), extending the lifetime of single qubit logical states using a seven qubit truncation of the code (see their fig. 1). That article gives the stabilizers, composite logical operators, and composite states in their eqns. (1)–(5). The figure below primitively cut/pastes from their fig. 1, and extends it to (a) 10 qubits, and (b) 13 qubits:

![Diagram](image)

(a) i) After implementing the stabilizer projections, how many logical qubits are encoded in figure (a)? ii) Write down logical operators $X_i$ and $Z_i$ for this number of qubits, and indicate why they satisfy the correct commutation relations. (It might be easiest just to draw them on the figure, as in other figures used in the lec26 notes, since the qubit labelling is non-standard.) iii) Write down the (6-qubit) states that correspond to the $+1$ projections of the four stabilizer operators. Identify those states as logical kets by their $Z_i$ eigenvalues $(-1)^x$ (i.e., generalizing eqns. (4,5) in the article).

b) i) After implementing the stabilizer projections, how many logical qubits are encoded in figure (b)? ii) Write down logical operators $X_i$ and $Z_i$ for this number of qubits, and indicate why they satisfy the correct commutation relations. (Again perhaps easiest just to draw them on the figure, due to the non-standard qubit labelling.) iii) Write down the (8-qubit) states that correspond to the $+1$ projections of the five stabilizer operators. Identify those states as logical kets by their $Z_i$ eigenvalues $(-1)^x$ (i.e., generalizing eqns. (4,5) in the article).
Problem 5: Phase shift gates

A. Show that the circuit in Fig. 28.1 (last page) of Aaronson notes has the property depicted. What operators $S$ give the desired output state $T|\psi\rangle$ (up to an overall phase) for the two possible measurement values 0,1 of the lower qubit?

It turns out possible to create the state $|\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$ reliably for $\theta = \pi/4$: $|m\rangle = |0\rangle + e^{i\pi/4}|1\rangle$, a so-called “magic state”. This state can then be used to create the gate $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$, with $T^4 = Z$ (as considered in ps2#5), which together with $H$ and cNOTs provides a universal gate set.

B. Given access to a state of the form $|\theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$, it’s also possible to implement the phase shift operator $A_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, via the two circuits below:

![Circuit Diagram 1](image1)

i) The circuit at left takes as input a state $|\psi\rangle$ and ancillary state $|\theta\rangle$, first implements a joint $Z \otimes Z$ measurement on the two qubits, and then an $X$ measurement on the ancillary qubit. Show that if both measurements give 0, then the output state is $A_\theta|\psi\rangle$.

ii) In the circuit at right above, the measurements are connected to classically operated gates to make corrections if they’re non-zero. What operators $O_1, O_2$ will result in the correct phase shift gate $A_\theta$ (up to an overall phase)?

[Note that for the $O_i$ you’re permitted to use operators of the form $\exp(i\alpha Z)$, even though they may be implementable in practice only for certain values of $\alpha$.]

---

1 Look for “magic state distillation”, a procedure which uses error correction-like codes to “distill” a higher fidelity version of a state from multiple lower fidelity versions.

2 i.e., combinations of which can be used to realize any multi-qubit unitary transformations. This is how the gates are currently expected to be realized in a fault tolerant manner in practice, with implementations designed for use in the surface code.

3 Note that this operator is equal to $e^{i\frac{\theta}{2}} e^{-i\frac{\theta}{2} Z}$.

4 This projects onto one of the $\pm 1$ eigenspaces of the joint operator $Z \otimes Z$.

5 projecting onto one of the $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ eigenspaces of the $X$ operator.

6 The procedure effectively ‘teleports’ the phase shift from a canonical state $|\theta\rangle = A_\theta|0\rangle$ to an arbitrary state $|\psi\rangle$. 

---

| Page 4 |
There’s a generalization of part A to commonly used multi-qubit operators. A “Pauli operator” \( P \) is an operator drawn from \( \{ I, X, Y, Z \} \otimes^n \), i.e., a direct product of \( I, X, Y, Z \) matrices. For example the stabilizers in the 5 and 7 qubit codes are all Pauli operators.\(^7\) These operators\(^8\) all square to 1, so have eigenvalues \( \pm 1 \). Their action on a multi-qubit state \(| \Psi \rangle = \alpha_0 |\Psi\rangle_0 + \alpha_1 |\Psi\rangle_1 \) is given by \( P |\Psi\rangle = \alpha_0 |\Psi\rangle_0 - \alpha_1 |\Psi\rangle_1 \) where \(|\Psi\rangle_x \) are in \((-1)^x = \pm 1\) eigenspaces of \( P \).\(^9\) The operator \( P_\theta |\Psi\rangle = \alpha_0 |\Psi\rangle_0 + e^{i\theta} \alpha_1 |\Psi\rangle_1 \), directly generalizes the above \( A_\theta \) from the 1-qubit case, with \( P = Z \). \( P_\theta \) can be realized via the circuits below:

i) The circuit at left above takes as input a state \(| \Psi \rangle \) and the ancillary state \(| \theta \rangle \), first implements a joint \( P \otimes Z \) measurement on the \( n + 1 \) qubits, and then an \( X \) measurement on the ancillary qubit. Show that if both measurements give 0, the output state is \( P_\theta |\psi\rangle \).

ii) In the circuit at right above, the measurements are connected to classically operated gates to make corrections if they don’t measure to zero. What operators \( O_1, O_2 \) will result in the correct phase shift gate (up to an overall phase)? [Note that for the \( O_i \) you’re permitted to use operators of the form \( \exp i \alpha P \), even though they may be implementable in practice only for certain values of \( \alpha \).]

Using the 1-qubit magic state \( |m\rangle = |0\rangle + e^{i\pi/4} |1\rangle \), it is thus possible to robustly implement essential \( P_{\pi/4} \) operators (satisfying \( P_{\pi/4}^4 = P \), analogous to \( T^4 = Z \)).

---

\(^7\) e.g., \( M_0 = Z \otimes X \otimes X \otimes Z \otimes I \) for the 5-qubit code, or \( N_1 = I \otimes Z \otimes I \otimes Z \otimes I \otimes Z \otimes Z \) in the 7-qubit code, and so on.

\(^8\) They are also closed under multiplication (including a factor of \( \pm 1, \pm i \)) and form a group known as the Pauli group.

\(^9\) Note that \( P \) should not be confused with a projection operator. In the notation used in lecture, the projection operators \( P_x^P = (1 + (-1)^x P) / 2 \) satisfy \( P_x^P |\Psi\rangle = \alpha_x |\Psi\rangle_x \).
Problem 6: [Bonus] 3-qubit state tomography for GHZ

To verify that they produce the state $|\Phi\rangle$ reliably, the authors of arXiv:1004.4324 perform “3-qubit state tomography”, a generalization of the 1-qubit state tomography considered earlier (problem #7 of problem set 3). They say they can “prepend sets of single-qubit rotations to the readout pulse. The rotations consist of all combinations of $I, R_x^\pi, R_x^{\pi/2}, \text{and } R_y^{\pi/2}$ on the three qubits.” These means they can apply independently any of $1, R_x^\pi = -iX, R_x^{\pi/2} = \frac{1}{\sqrt{2}}(1 - iX)$, and $R_y^{\pi/2} = \frac{1}{\sqrt{2}}(1 - iY)$ before measuring the three qubits, a total of $4^3 = 64$ possibilities.

a) We can define the vector of operators $\vec{P} = \{1, X, Y, Z\}$, and use its expectation values $\vec{p} = \langle \psi | \vec{P} | \psi \rangle$ to characterize any 1-qubit state $|\psi\rangle$ as in problem 3 #7. What are the values of $\langle \psi | \vec{P} | \psi \rangle$ for the states $|\psi\rangle = |0\rangle$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$? In the earlier problem, a specific set of measurements permitted determining the expectation values $\langle \psi | X | \psi \rangle$ and $\langle \psi | Z | \psi \rangle$ for the general 1-qubit state. How can the last three components of $\vec{p}$ be determined in terms of applying some subset of the operators $(I, R_x^\pi, R_x^{\pi/2}, R_y^{\pi/2})$ to the state $|\psi\rangle$ before measurement?

b) State tomography can be extended to characterize 2-qubit states $|\Psi\rangle$, by considering the sixteen 2-qubit operators $\vec{P}_{(2)} = \{1, X, Y, Z\}^\otimes 2$ (i.e., operators of the form $1 \otimes 1, 1 \otimes X, 1 \otimes Y, \ldots, X \otimes 1, \ldots, X \otimes Y, \ldots, Z \otimes Z$). The expectation values are again expressed as $\vec{p} = \langle \Psi | \vec{P}_{(2)} | \Psi \rangle$. What values of the $p_i$ characterize the 2-qubit Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (i.e., which of the expectation values $\langle \Psi | \vec{P}_{(2)} | \Psi \rangle$ will be non-zero in that state)? Which linear combinations* of measurements determine those values (i.e., measurements after applying some of $I, R_x^\pi, R_x^{\pi/2}, R_y^{\pi/2}$ to one or the other qubit, or to both)?

c) Finally, 3-qubit states are characterized by the expectation values of the sixty-four 3-qubit operators $\vec{P}_{(3)} = \{1, X, Y, Z\}^\otimes 3$ (i.e., operators of the form $1 \otimes 1 \otimes 1, X \otimes 1 \otimes 1, \ldots, X \otimes Y \otimes 1, \ldots, Z \otimes Z \otimes Z$). Which of the expectation values $\langle \Phi | \vec{P}_{(3)} | \Phi \rangle$ will be non-zero in the 3-qubit “GHZ” state $|\Phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ in problem 2 #7? Which linear combinations of measurements determine those values (i.e., measurements after applying some of $I, R_x^\pi, R_x^{\pi/2}, R_y^{\pi/2}$ to one or more of the qubits)?

* For example, $\langle \Psi | Z \otimes Z | \Psi \rangle = \sum_{x,y} \langle \Psi | Z \otimes Z | xy \rangle \langle xy | \Psi \rangle = \sum_{x,y} (-1)^{x+y} |\langle xy | \Psi \rangle|^2 = \sum_{x,y} (-1)^{x+y} p_{\Psi}(x,y)$ (where $p_{\Psi}(x,y)$ is the probability of measuring $x,y$ for the two qubits in the state $|\Psi\rangle$), so $\langle \Psi | Z \otimes Z | \Psi \rangle$ is estimated by $(N_{00} - N_{01} - N_{10} + N_{11})/N$, where $N_{xy}$ is the number of times the result $xy$ is measured in a total of $N = \sum_{x,y=0}^1 N_{xy}$ trials. Similarly, $\langle \Psi | X \otimes Z | \Psi \rangle = \langle \Psi | (R_y^{\pi/2} Z R_y^{\pi/2}) \otimes Z | \Psi \rangle = \langle \Psi | (R_y^{\pi/2} \otimes I)(Z \otimes Z)(R_y^{\pi/2} \otimes I)| \Psi \rangle = \sum_{x,y} \langle \Psi | (R_y^{\pi/2} \otimes I)(Z \otimes Z)| xy \rangle \langle xy | (R_y^{\pi/2} \otimes I)| \Psi \rangle = \sum_{x,y} (-1)^{x+y} |\langle xy | R_y^{\pi/2} \otimes I | \Psi \rangle|^2 \approx \sum_{x,y} (-1)^{x+y} N_{xy}/N$, where again $N_{xy}$ is the number of times the result $xy$ is measured in a total of $N = \sum_{x,y=0}^1 N_{xy}$ trials, but now $R_y^{\pi/2}$ is always applied to the left qubit before measurements.