\[ |x\rangle \langle x| \quad \psi \rangle \langle \psi | \]

\[ M = \sum_i \psi_i \langle \psi_i | \psi \rangle \]

\[ M|\psi\rangle = \sum_i \psi_i |\psi_i\rangle \]

\[ \langle i | M | j \rangle = M_{ij} \]

\[ M = |x\rangle \langle y| \]

\[ M_{ij} = \langle i | x \rangle \langle y | j \rangle = \delta_{ix} \delta_{yj} \]
\[ |x \rangle \langle y | = \begin{pmatrix} 0 \\ \ddots \\ 0 \\ 1 \\ 0 \\ \ddots \\ \ddots \\ 0 \end{pmatrix} \]

\[ P_x = |x \rangle \langle x | \]

\[ |\psi \rangle = \sum_x a_x |x \rangle \quad P_x |\psi \rangle = a_x |x \rangle \]

\[ |\psi \rangle \langle \psi | = \begin{pmatrix} 0 \\ \ddots \\ 0 \\ 1 \\ 0 \\ \ddots \\ \ddots \\ 0 \end{pmatrix} \]

\[ \sum_x \frac{1}{2^n} |x \rangle \langle x | \leq |y \rangle \langle y | \leq \sum_x \frac{1}{2^n} \]
Collision problem

N items, classically check m

\[ N \sim \frac{1}{2} m(m-1) \quad \text{so need} \quad m \sim N^{1/2} \]

Quantum: look at some fraction \( m \sim N^a \), call them marked

Grover \( \sqrt{\frac{N}{m}} \sim (\frac{N}{N^a})^{1/2} \)

Optimum \( N^a \sim N^{(1-a)/2} \)

\[ a = 1 - \frac{a}{2} \Rightarrow a = 1/3 \]

So both \( N^{1/3} \)
Parity of n bits: $QM \frac{n}{2}$ (barely)

OR (any one is on): No speedup

Why $\sqrt{N}$?

Classically work with probabilities
\[ \frac{1}{\sqrt{N}}, \frac{2}{\sqrt{N}}, \ldots, \frac{m}{\sqrt{N}} \sim 1 \]

$QM$: work with amplitudes
\[ \frac{1}{\sqrt{N}}, \frac{2}{\sqrt{N}}, \frac{3}{\sqrt{N}}, \ldots, \frac{T}{\sqrt{N}} \sim 1 \]
Quantum Zeno

(like the $\Rightarrow$ polarization)

$|1\rangle = \cos 3\pi/5 |0\rangle + \sin 3\pi/5 |1\rangle$

$|1\rangle = |0\rangle = -\sin 3\pi/5 |0\rangle + \cos 3\pi/5 |1\rangle$

Start with $|10\rangle$, measure $|1\rangle$, $|1w\rangle$

$P(1\rangle) = |K| |1\rangle|^2 = \cos^2 3\pi/5 \approx 1 - 3^2$

$P(1\rangle) \approx 3^2$

Repeat $N = \frac{11/2}{3}$ times so $N \approx 11/2$

prob of at least one $|1w\rangle$

$\leq \text{prob of all } |1w\rangle = O(N\varepsilon^2) = O(3)$.

(union bound) So essentially always measure $|1\rangle$, and after $N$ measurements state is $|11\rangle$. 

Watched Pot

Quantum state keeps drifting/rotating if left alone. Instead keep measuring in |0\rangle, |1\rangle basis.

Let’s say it would rotate by $\theta$ in each timestep

\[
\begin{align*}
P(|0 \rangle) &= 1 - \theta^2 \\
P(|1 \rangle) &= \theta^2
\end{align*}
\]

So stays at $|0\rangle$ w/ high prob.

Repeat $O(1/\theta)$ steps, and probability to be $|1\rangle$ is $O(\theta)$.

But without measuring, would have drifted to $|1\rangle$ in $1/2\theta$ steps.
Elitzur-Vaidman Bomb
has input bit \( b = 0 \) don't query
\( b = 1 \) query, always
sets it off

Quantum:
Let \( |b\rangle = \alpha |10\rangle + \beta |11\rangle \)

- \( B? \) -
  if no bomb, nothing happens
  return \( |b\rangle \)
  if bomb, measure in \( |0\rangle |1\rangle \)
basis:

\[
|0\rangle \rightarrow \text{no query}
\]

\[
|1\rangle \rightarrow \text{Explosion}
\]

Start with \( |b\rangle = 0 \), apply
a sequence of \( R_3 = \begin{pmatrix} \cos 3 & -\sin 3 \\ \sin 3 & \cos 3 \end{pmatrix} \)
first \( |b\rangle = (\cos \theta |0\rangle + \sin \theta |3\rangle) \)

Repeat \( \frac{11}{2} \times 3 \) times:

Total probability of exploding = \( \prod \frac{3^2}{2^2} \approx \frac{3^3}{2^3} \)

Then measure:

if \( |10\rangle \) there's a bomb (watched pot)

if \( |11\rangle \) no bomb (drifted)

(requires bomb that can be quantum interrogated)
Quantum Teleportation: Alice, Bob share a Bell state \( \Phi_{ab} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \).

Alice also has some (unknown) \(|\psi\rangle\).

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]  

(See next page) To recover \(|\psi\rangle\),

If Alice measures:

\( x, y = 0, 0 \)

0, 1

1, 0

1, 1

Bob needs to apply:

\( \text{I} \)

\( x \)

\( z \)

\( x \)
\[ |\Psi_{ab}\rangle = (\alpha |10\rangle + \beta |11\rangle) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]

Apply CNOT \[ \langle 0\rangle \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle) + \beta |1\rangle \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle) \]

\[ \frac{1}{\sqrt{2}} |0\rangle - |1\rangle \]

To correct Bob needs to apply 

\[ \frac{1}{\sqrt{2}} |10\rangle (\alpha |10\rangle + \beta |11\rangle) \]

\[ \frac{1}{\sqrt{2}} |11\rangle (\alpha |10\rangle - \beta |11\rangle) \]

\[ \frac{1}{\sqrt{2}} |10\rangle (\alpha |11\rangle - \beta |10\rangle) \]

\[ \frac{1}{\sqrt{2}} |11\rangle (\alpha |11\rangle - \beta |10\rangle) \]

\[ |\Psi\rangle_b \]