The Quantum Fourier Transform $U_{\rm FT}$ is defined to transform the state $|x\rangle$ according to

$$U_{\rm FT}|x\rangle = \frac{1}{2^{n/2}} \sum_{0\le y<2^n} e^{2\pi i xy/2^n}|y\rangle\ , \tag{1}$$

where xy is the ordinary product of the numbers x and y.

In terms of the binary expansion $x = x_{n-1}\ldots x_1x_0$, division by 2 moves the decimal point to the left by one place value, so $x/2^n = 0.x_{n-1}\ldots x_1x_0$ (recall that in binary, $.1 = 1/2$, $.01 = 1/4$, $.11 = 3/4$, $x_0/8 = .00x_0$, and so on). In terms of its binary expansion, $y = y_{n-1}\ldots y_1y_0$, the numerical value of y is given by $y = \sum_{j=0}^{n-1} y_j 2^j$. Since multiplication by 2^j just shifts the decimal point to the right by j places, we can write[1]

$$\begin{aligned} xy/2^n = y_{n-1}\cdot x_{n-1}\ldots x_1.x_0 \ +\ y_{n-2}\cdot x_{n-1}\ldots x_2.x_1x_0\ +\ \cdots \\ +\ y_1\cdot x_{n-1}.x_{n-2}\ldots x_0\ +\ y_0\cdot 0.x_{n-1}\ldots x_0\ . \end{aligned}$$

Integer multiples of 2π in the exponent of $e^{2\pi i xy/2^n}$ do not contribute to the phase, so we can retain only

$$e^{2\pi i xy/2^n} = e^{2\pi i y_{n-1} 0.x_0}\, e^{2\pi i y_{n-2} 0.x_1x_0}\cdots e^{2\pi i y_1 0.x_{n-2}\ldots x_0}\, e^{2\pi i y_0 0.x_{n-1}\ldots x_0}\ .$$

Since y_j indicates whether the $j^{\rm th}$ bit of y (counting from the right) is 1 or 0, it follows that the formula (1) for $U_{\rm FT}$ can be written:

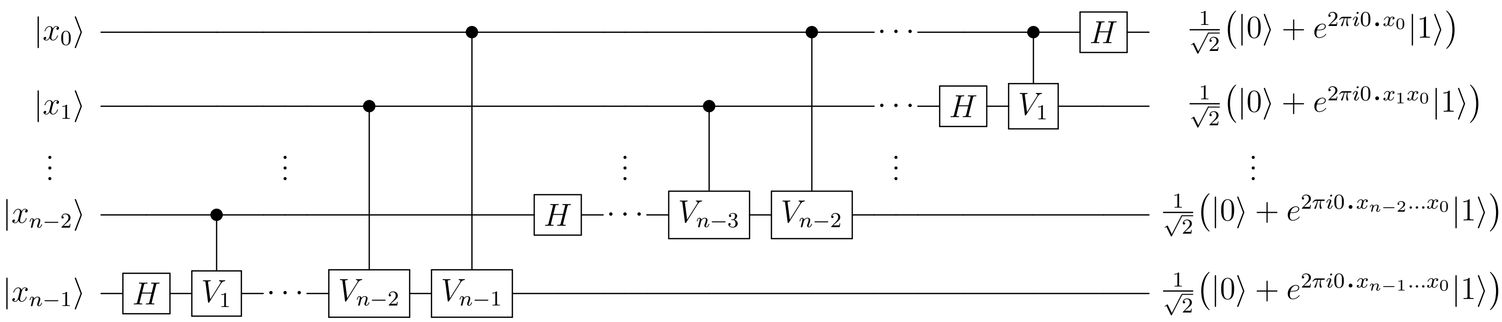
$$U_{\rm FT}|x_{n-1}\ldots x_0\rangle = \frac{1}{2^{n/2}}\big(|0\rangle + e^{2\pi i 0.x_0}|1\rangle\big)\big(|0\rangle + e^{2\pi i 0.x_1x_0}|1\rangle\big)\cdots\big(|0\rangle + e^{2\pi i 0.x_{n-1}\ldots x_0}|1\rangle\big) \tag{2}$$

(where the sum over the two values of each qubit generates the sum over all y in (1)).

To draw a circuit diagram for this unitary transformation of states, we define the phase operator $V_k \equiv \begin{pmatrix}1 & 0\\ 0 & e^{\pi i/2^k}\end{pmatrix}$. Then for example $H|x_0\rangle = \frac{1}{\sqrt2}(|0\rangle + (-1)^{x_0}|1\rangle) = \frac{1}{\sqrt2}(|0\rangle + e^{2\pi i 0.x_0}|1\rangle)$, and $V_1^{x_0}H|x_1\rangle = V_1^{x_0}\frac{1}{\sqrt2}(|0\rangle + e^{2\pi i 0.x_1}|1\rangle) = \frac{1}{\sqrt2}(|0\rangle + e^{2\pi i 0.x_1x_0}|1\rangle)$, since $V_1^{x_0}$ only adds the additional phase $2\pi i/4 = 2\pi i\cdot 0.01$ if $x_0 = 1$. Eqn. (2) can thus be rewritten

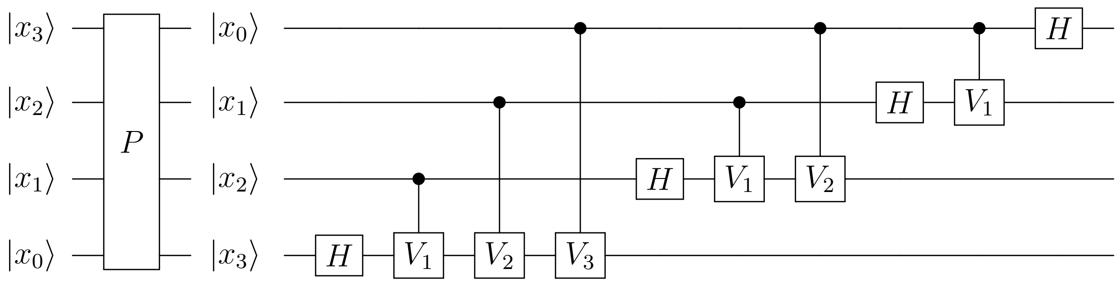
$$\begin{aligned} U_{\rm FT}|x_{n-1}\ldots x_0\rangle = (H|x_0\rangle)\big(V_1^{x_0}H|x_1\rangle\big)\big(V_2^{x_0}V_1^{x_1}H|x_2\rangle\big)\cdots \\ \big(V_{n-2}^{x_0}V_{n-3}^{x_1}\cdots V_1^{x_{n-3}}H|x_{n-2}\rangle\big)\big(V_{n-1}^{x_0}V_{n-2}^{x_1}\cdots V_1^{x_{n-2}}H|x_{n-1}\rangle\big)\ , \end{aligned} \tag{3}$$

which provides the circuit for $U_{\rm FT}$:



[1] In base 10, this corresponds to, e.g., $329\cdot125/10^3 = (300\cdot125+20\cdot125+9\cdot125)/10^3 = 3\cdot12.5+2\cdot1.25+9\cdot.125$

Note that there is one H and at most $n-1$ controlled-V's for each qubit, so the number of gates grows at most quadratically in n. Notice also that the expansion of $xy/2^n$ couples the least significant $|x_0\rangle$ to the most significant $|y_{n-1}\rangle$, and so on, $|x_j\rangle$ to $|y_{n-1-j}\rangle$. The output qubits in the above figure are drawn according to the usual convention that the most significant qubit is at the top. To retain this convention as well for the input qubits, we insert a permutation operator to reorder them appropriately (in a physical realization, this is just a question of how the "wires" are connected to the gates). For the explicit case $n = 4$, the above circuit diagram becomes



$$\frac{1}{2^2}\sum_{0\le y<2^4} e^{2\pi i xy/2^4}|y\rangle$$