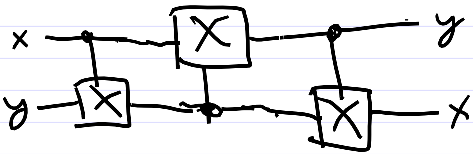
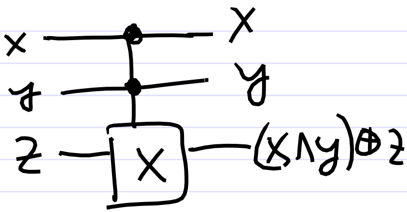
Lecture 3, 10 Sep 2020

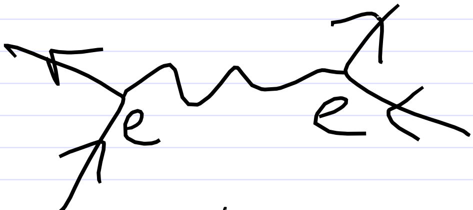
See added slides 9, 10, 12 to lec2

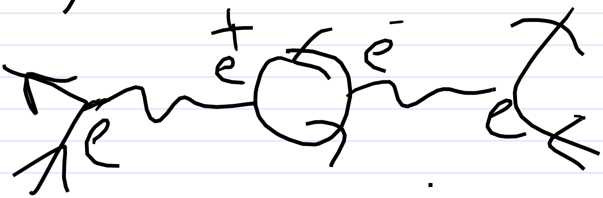
$S_{ij} =$



$T_{xyz} =$

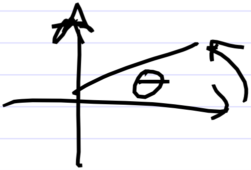






2d rotations $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$R(\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$



$\det R(\theta) = \underline{1}$

$Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\quad \det Z = \underline{-1}$

$Z R(\theta) = R(-\theta) Z$

(if odd # of Z's)

$\cancel{Z}R(\theta_1) Z R(\theta_2) Z R(\theta_3) \cdots = \underline{R(2\theta) Z}$

$2\theta = \theta_1 - \theta_2 + \theta_3 \cdots$ $\quad = \underline{R(\theta) Z R(-\theta)}$

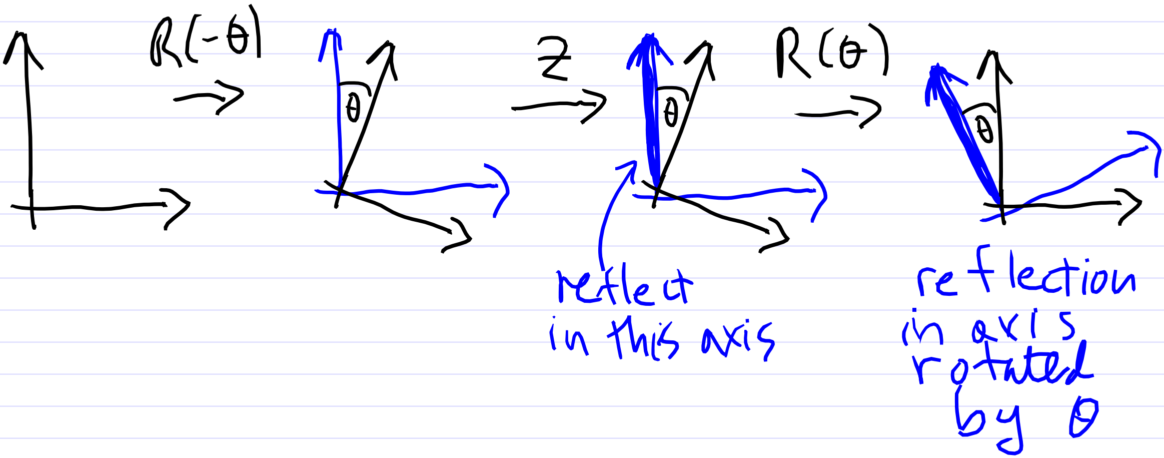
So only two possibilities:
1) even # Z's, det = 1
rotation
2) odd # Z's, det = −1
reflection in rotated axis

i.e. Reflection about axis rotated by θ
(see next slide)

$$\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix}c & -s\\ s & c\end{pmatrix} = \begin{pmatrix}c & -s\\ -s & -c\end{pmatrix} = Z R(\theta)$$

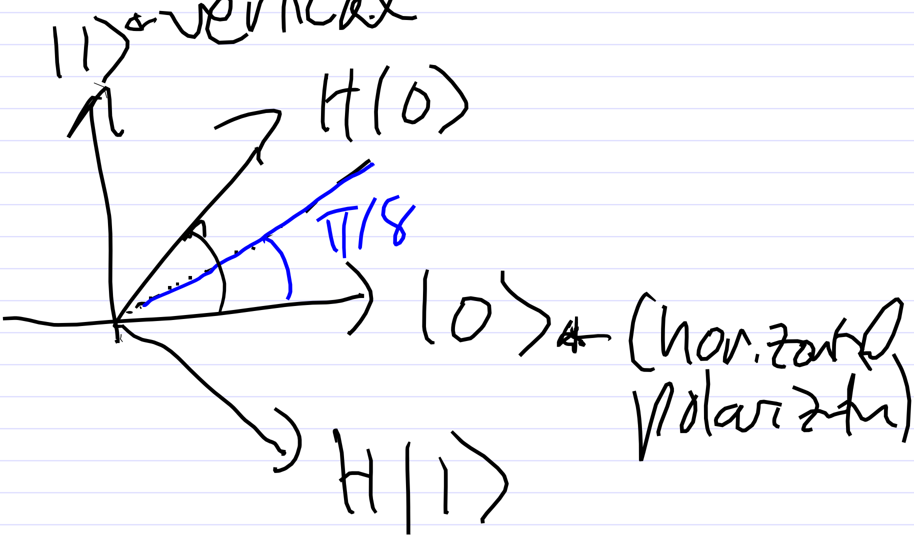
$$\begin{pmatrix}c & s\\ -s & c\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix} = \begin{pmatrix}c & -s\\ -s & -c\end{pmatrix} = R(-\theta) Z$$

Action of $R(\theta) Z R(-\theta)$



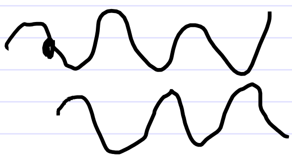
$$H = \frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \det H = -1$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



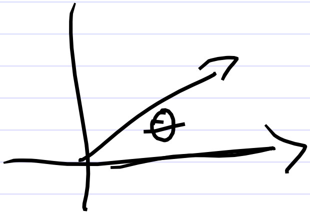
photon, how to implement 1/2 wave plate. bi-refringent

↳ different indices of refraction

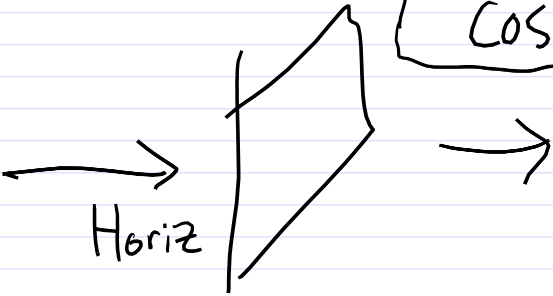


$\frac{1}{2}\lambda \Leftrightarrow A \to -A$

linear polarization



$$\boxed{\begin{gathered}\cos\theta|0\rangle + \sin\theta|1\rangle \\ \cos^2\theta = p(\theta)\end{gathered}}$$



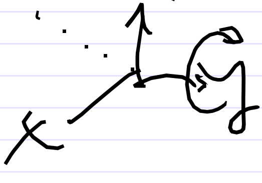
polarizer

Amplitude of light
$\sim \cos\theta$

Intensity
$\sim \cos^2\theta$

(Someone asked about implementing Hadamard in superconducting qubit)

$$e^{i\sigma_y \theta/2}$$



1 Qubit

$\alpha_0, \alpha_1 \in \mathbb{C}$

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \qquad |\alpha_0|^2 + |\alpha_1|^2 = 1$$

like cbit $\alpha_0 = 1$ $\alpha_1 = 0$

2 Qubit

$$|\Psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\sum |\alpha_{ij}|^2 = 1$$

Measure $|\psi\rangle$

1-qubit

$$|\psi\rangle \xrightarrow{\boxed{M}\,x} |x\rangle \qquad p(x) = |\alpha_x|^2$$

0 $|\alpha_0|^2$

1 $|\alpha_1|^2$

n-qubit

$$|\Psi\rangle = \sum_{x=0}^{2^n-1} \alpha_x |x\rangle$$

2^n amplitudes

α_x, $\sum_x |\alpha_x|^2 = 1$

$$\xrightarrow{\boxed{M}\,x} |x\rangle \qquad p(x) = |\alpha_x|^2$$

5 qubit e.g. $|10110\rangle$ $\quad P(|10110\rangle) = |\alpha_{10110}|^2$

if $\neq 0$

$$|\psi\rangle = \begin{pmatrix} \alpha_{00\ldots0} \\ \alpha_{0\ldots1} \\ \vdots \\ \alpha_{11\ldots10} \\ \alpha_{11\ldots11} \end{pmatrix}$$

Cbit states = "computational basis"

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entanglement

$$|\Psi\rangle_2 = |\psi\rangle \otimes |\varphi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

$$\begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

product state iff $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

$$\frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= H \otimes H\, |11\rangle$$

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \lambda = \begin{matrix} 0 \\ 1 \end{matrix}$$

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

entangled

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

α_{ij}

in general SVD* of amplitudes α:

$$\lambda = \pm\frac{\sqrt{2}}{2}$$

Unentangled iff a single singular value is non-zero

*a.k.a. Schmidt decomposition

Appendix I (linear algebra)

$$|\psi\rangle = \sum_{x=0}^{N} \alpha_x |x\rangle$$

$$|\varphi\rangle = \sum_{x=0}^{N} \beta_x |x\rangle$$

$\alpha|\psi\rangle + \beta|\varphi\rangle$ is also a vector

inner product $\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$

$\langle\varphi|\varphi\rangle > 0$ if $|\varphi\rangle \neq 0$

"Anti-linear": $\langle\varphi|\alpha\psi_1 + \beta\psi_2\rangle = \alpha\langle\varphi|\psi_1\rangle + \beta\langle\varphi|\psi_2\rangle$

In components $\langle\varphi|\psi\rangle = \sum_x \beta_x^* \alpha_x$

$\langle\psi|\psi\rangle = \sum_x |\alpha_x|^2$

A linear transformation satisfies

$$A(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha A|\psi\rangle + \beta A|\varphi\rangle$$

If it preserves the norm of vectors

$\langle A\psi|A\psi\rangle = \langle\psi|\psi\rangle$, then called "Unitary"

(takes unit vectors to unit vectors)

Unitary U also preserves arbitrary inner products

$$\langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle$$
$$= \langle \varphi | U^\dagger U | \psi \rangle \quad \text{so } U^\dagger U = \mathbb{1} = U U^\dagger$$

and $U^\dagger = U^{-1}$ where $U^\dagger = (U^T)^*$

i.e. $(U^\dagger)_{ij} = U^*_{ji}$ $\qquad U_{ij} = \langle i | U | j \rangle$

Consider components in some basis

$\alpha_i = \langle i | \psi \rangle$, $\beta_j = \langle j | \varphi \rangle$

See how this works:

$$\sum_i \beta_i^* \alpha_i = \sum_{i,j,k} (U_{ij}\beta_j)^* U_{ik}\alpha_k = \sum_{ijk} \beta_j^* \underbrace{U^*_{ij} U_{ik}}_{\text{has to be } \delta_{jk}} \alpha_k$$

So $\sum_i U^*_{ij} U_{ik} = \delta_{jk}$

$$= \sum_i (U^\dagger)_{ji} U_{ik} \quad \text{or } U^\dagger U = \mathbb{1}$$

Slighty different from Hermitian, $H^\dagger = H$, but a matrix can be both unitary and Hermitian, e.g., Hadamard, Z, X

Appendix II

consider 2x2 unitary matrices, the group $U(2)$

$$u = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \qquad u^\dagger u = \mathbb{1} \Rightarrow u = e^{i\varphi/2}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix}$$

8 real params,
4 conditions

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\det u = e^{i\varphi}$$

$$\varphi = 0 \Rightarrow SU(2)$$

write

$$u = u_0 \mathbb{1} + i\vec{v}\cdot\vec{\sigma}$$

$$= \begin{pmatrix} u_0 + iv_3 & iv_1 + v_2 \\ iv_1 - v_2 & u_0 - iv_3 \end{pmatrix}$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$= \left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right)$$

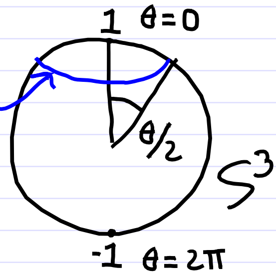
$$\det u = u_0^2 + \vec{v}^2 = 1$$ (3 sphere S^3 embedded in $\mathbb{R}^4$)

check $$uu^\dagger = (u_0\mathbb{1} + i\vec{v}\cdot\vec{\sigma})(u_0\mathbb{1} - i\vec{v}\cdot\vec{\sigma})$$
$$= (u_0^2 + v^2)\mathbb{1} = \mathbb{1}$$

use polar coordinates

$$u_0 = \cos\frac{\theta}{2} \qquad \vec{v} = \sin\frac{\theta}{2}\,\hat{n}$$

unit vector in 3d on S^2



so finally $U = u_0 \mathbb{1} + i\vec{v}\cdot\vec{\sigma}$

$$= \cos\frac{\theta}{2}\,\mathbb{1} + i\sin\frac{\theta}{2}\,\hat{n}\cdot\vec{\sigma}$$

$$= e^{i(\hat{n}\cdot\sigma)\frac{\theta}{2}}$$

(where from piazza post:)

[PG] Note that a matrix exponential is defined in terms of its power series expansion, so in general
$\exp i(\hat{n}\cdot\sigma)x = I + i(\hat{n}\cdot\vec{\sigma})x + (i(\hat{n}\cdot\vec{\sigma})x)^2/2 + (i(\hat{n}\cdot\vec{\sigma})x)^3/3! + (i(\hat{n}\cdot\vec{\sigma})x)^4/4! + \ldots$
(where $\hat{n}$ is a unit vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$).
Using $\sigma_i\sigma_j = I\delta_{ij} + i\epsilon_{ijk}\sigma_k$, we see that $(\hat{n}\cdot\vec{\sigma})^2 = n_i n_j I\delta_{ij} = (\hat{n}\cdot\hat{n})I = I$, and the above becomes

$$\begin{aligned}\exp i(\hat{n}\cdot\sigma)x &= I + i(\hat{n}\cdot\vec{\sigma})x - Ix^2/2 - i(\hat{n}\cdot\vec{\sigma})x^3/3! + Ix^4/4! + \ldots\\ &= I(1 - x^2/2 + x^4/4! - \ldots) + i\hat{n}\cdot\vec{\sigma}(x - x^3/3! + \ldots)\\ &= I\cos(x) + i\hat{n}\cdot\vec{\sigma}\sin(x)\end{aligned}$$

i.e., since $(\hat{n}\cdot\vec{\sigma})^2 = \mathbb{1}$, $i\hat{n}\cdot\vec{\sigma}$ plays the role of i

generalizing $e^{ix} = \cos x + i\sin x$

where $e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! + \ldots$

$$= \sum_{n=0}^{\infty} x^n/n!$$

Appendix III

$\sigma_x, \sigma_y, \sigma_z$ basis for traceless 2×2 Hermitian matrices

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\mathrm{tr}\, A = \sum_k A_{kk}$
satisfies $\mathrm{tr}\, AB = \mathrm{tr}\, BA$

$$\sigma_x \sigma_y = i\sigma_z, \quad \sigma_y \sigma_z = i\sigma_x, \quad \sigma_z \sigma_x = i\sigma_y; \quad \sigma_i^2 = \mathbb{1}$$

more compactly:

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i\varepsilon_{ijk}\sigma_k$$

$\mathrm{tr}\, \sigma_i \sigma_j = 2\delta_{ij}$ $\quad\quad$ $\varepsilon_{xyz} = 1 = -\varepsilon_{yxz} = \ldots$

Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$$\vec{a} \cdot \vec{\sigma} = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z$$

$$= \begin{pmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{pmatrix}$$

Now from

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = a_i b_j \sigma_i \sigma_j$$
$$= a_i b_j (\delta_{ij} \mathbb{1} + i\varepsilon_{ijk}\sigma_k)$$
$$= \vec{a}\cdot\vec{b}\,\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

(where $(\vec{a}\times\vec{b})_i = \varepsilon_{ijk} a_j b_k$)

Since the σ_i are traceless,

$$\mathrm{tr}\left[(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})\right] = \vec{a}\cdot\vec{b}\,\mathrm{tr}\,\mathbb{1} = 2\,\vec{a}\cdot\vec{b}$$

Now consider action of Unitary U

$$U(\vec{a}\cdot\vec{\sigma})U^\dagger = \vec{a}'\cdot\vec{\sigma}$$
$$U(\vec{b}\cdot\vec{\sigma})U^\dagger = \vec{b}'\cdot\vec{\sigma}$$

$\mathrm{tr}\,UAU^\dagger = \mathrm{tr}\,U^\dagger UA = \mathrm{tr}A$

(still Traceless Hermitian)

$(UAU^\dagger)^\dagger = (U^\dagger)^\dagger A^\dagger U^\dagger = UAU^\dagger$

but $\mathrm{tr}\left[U(\vec{a}\cdot\vec{\sigma})U^\dagger U(\vec{b}\cdot\vec{\sigma})U^\dagger\right] = \vec{a}'\cdot\vec{b}'$

(cyclicity of trace and $U^\dagger U = 1$) $= \mathrm{tr}\,(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b}$

so $\vec{a}\cdot\vec{b} = \vec{a}'\cdot\vec{b}'$ preserves dot products and the similarity transformation

$$U(\vec{a}\cdot\vec{\sigma})U^\dagger = \vec{a}'\cdot\vec{\sigma}$$
$$U(\vec{b}\cdot\vec{\sigma})U^\dagger = \vec{b}'\cdot\vec{\sigma}$$

induces a rotation $\vec{a}' = R_u\vec{a}$
$\vec{b}' = R_u\vec{b}$

of the 3d vectors $\vec{a}, \vec{b}$

Note that $\pm U$ induce the same rotation so $SO(3) \approx SU(2)/\mathbb{Z}_2$

What rotation is it?

Let $U = e^{i(\hat{n}\cdot\sigma)\frac{\theta}{2}}$ take $\vec{a} = \hat{n}$.

Then $U(\hat{n}\cdot\vec{\sigma})U^\dagger = \hat{n}\cdot\vec{\sigma}$

so must be a rotation about the $\hat{n}$-axis.

For simplicity, take $\hat{n} = \hat{z}$, so $\hat{n}\cdot\vec{\sigma} = \sigma_z$,

and consider action on $\vec{a} = \hat{x}$, so $\vec{a}\cdot\vec{\sigma} = \sigma_x$

$$e^{i\frac{\theta}{2}\sigma_z}\sigma_x e^{-i\frac{\theta}{2}\sigma_z}$$

$$= \left(\cos\frac{\theta}{2}\mathbb{1} + i\sigma_z\sin\frac{\theta}{2}\right)\sigma_x\left(\cos\frac{\theta}{2}\mathbb{1} - i\sigma_z\sin\frac{\theta}{2}\right)$$

$$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\sigma_x - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sigma_y$$

$$= \cos\theta\,\sigma_x - \sin\theta\,\sigma_y$$

$\vec{a} = (1, 0, 0)$

$\vec{a}' = (\cos\theta, -\sin\theta, 0)$

so $U(\hat{z},\theta) = \exp(i\sigma_z\theta/2)$ induces a clockwise rotation by θ about the $\hat{z}$-axis

and $U(\hat{n},\theta)$ a rotation by θ about the $\hat{n}$-axis

Can calculate $U(\hat{n}_2,\theta_2)U(\hat{n}_1,\theta_1) = U(\hat{n}_3,\theta_3)$