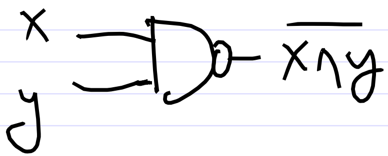
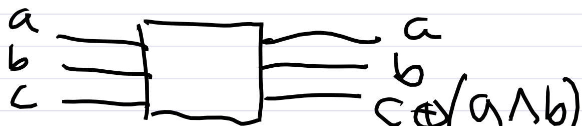
# Lecture 2 9 Sep 2020

## Reversible computing




| $x$ | $y$ | $\overline{x \wedge y}$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

## Tofolli gate




$c = 1$ outputs $\overline{a \wedge b}$

$b = c = 1$ outputs $\overline{a}$

$\oplus$ = addition mod 2

| $x$ | $y$ | $x \oplus y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

for quantum computing two qubit gates are sufficient!
(reversible classical needs 3-bit gates)

classical computing only erasure costs ~~en~~ energy!

<u>Bennett</u> = computing does not

Landauer '61 $kT\ln 2 = \Delta E$

$PV = nR_0T$ ideal gas constant

$R = N_A k_B$

$N_A = 6.023 \cdot 10^{23}$

$S = k_B \ln \Omega$

$I = \log_2 \Omega$

$\Omega = 2^n$

$= n$ bits

$dE = TdS$

$\Rightarrow \Delta S = k\ln 2 \Rightarrow \Delta I = \underline{1}$

$kT = \frac{1}{40}\,\text{eV} = .025\,\text{eV}$

$10^4$ erasures/sec $\Rightarrow \leq 10^{-11}$ W

1 cbit $|0\rangle, |1\rangle$

$\mathbb{1}: \begin{array}{l} |0\rangle \to |0\rangle \\ |1\rangle \to |1\rangle \end{array}$

$X: \begin{array}{l} |0\rangle \to |1\rangle \\ |1\rangle \to |0\rangle \end{array}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mathbb{1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\quad X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

# Tensor Product Spaces

$\vec{v} \in V^m \quad \alpha \vec{v}_1 + \beta \vec{v}_2 \in V^m$ $\quad \alpha, \beta \in \mathbb{R}, \mathbb{C}, \mathbb{Q}$

$\vec{w} \in W^n$

$\vec{v} \otimes \vec{w}$

$$\vec{v} = \sum_{i=1}^{m} \alpha_i \vec{e}_i$$

$$\vec{w} = \sum_{j=1}^{n} \beta_j \vec{f}_j$$

$$\vec{v} \otimes \vec{w} \in V^m \otimes W^n$$

$$= \sum_{i,j} \alpha_i \beta_j \vec{e}_i \otimes \vec{f}_j$$

$\vec{e}_i \otimes \vec{f}_j$

$(m \cdot n)$-dim basis

has components $(\alpha_i \beta_j)$

Operators $\quad M \quad \vec{v} \to M\vec{v}$

$\qquad N \quad \vec{w} \to N\vec{w}$

$$M \otimes N (\vec{v} \otimes \vec{w}) = (M\vec{v}) \otimes (N\vec{w})$$

$2$ Cbits $|00\rangle \ |01\rangle \ |10\rangle \ |11\rangle$

$4! = 24$ permutations

First operator is $S_{ij}$

$S_{01}|x\, y\rangle = |y\, x\rangle$ $\qquad S_{01} = S_{10}$

$|01\rangle \leftrightarrow |10\rangle$

$$S_{01} = S_{10} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

Swap operator

# 3 Qubits

$$\sum_{i,j,k} x_i \vec{e}_i \otimes y_j \vec{e}_j \otimes z_k \vec{e}_k$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \otimes \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_0 y_0 z_0 \\ x_0 y_0 z_1 \\ x_0 y_1 z_0 \\ x_0 y_1 z_1 \\ x_1 y_0 z_0 \\ x_1 y_0 z_1 \\ x_1 y_1 z_0 \\ x_1 y_1 z_1 \end{pmatrix}$$

$$\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$$

$$|5\rangle = |1\rangle|0\rangle|1\rangle = |101\rangle$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

binary expansion

$$x = \sum_{j=0}^{n-1} x_j 2^j$$

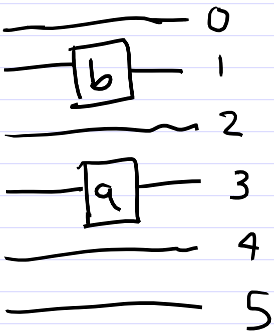
$$|x\rangle_n = |x_{n-1}\rangle \otimes |x_{n-2}\rangle \otimes \cdots \otimes |x_1\rangle \otimes |x_0\rangle$$

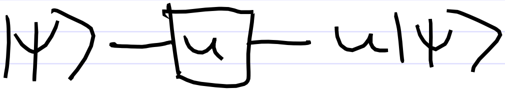
$$a\otimes b\,|xy\rangle = a|x\rangle \otimes b|y\rangle$$

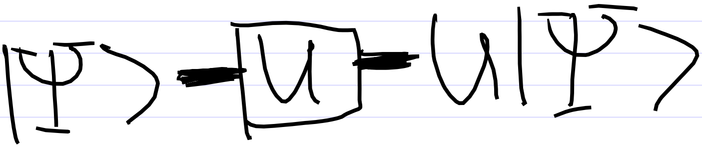
$$(a\otimes b)(c\otimes d) = (ac)\otimes(bd)$$

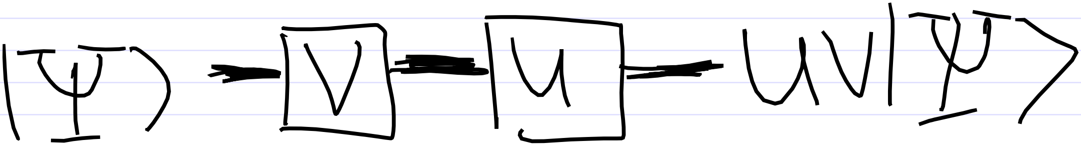
$$\overset{5}{\underline{1}}\otimes\overset{4}{\underline{1}}\otimes\overset{3}{a}\otimes\overset{2}{1}\otimes\overset{1}{b}\otimes\overset{0}{\underline{1}}$$ acting on 6 qubits

$$= a_3 b_1 =$$



CNOT = controlled-NOT

$C_i U_j$

control  target bit

if $i=0$, do nothing

if $i=1$, flip $j$

$$C_{10}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle$$

$$C_{01}|x\rangle|y\rangle = |x \oplus y\rangle|y\rangle$$

$$C_{10} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

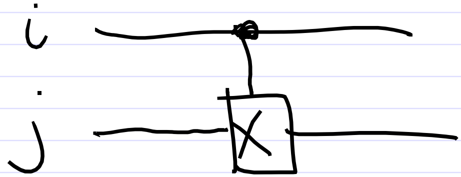
$|00\rangle, |01\rangle$

$|10\rangle \leftrightarrow |11\rangle$

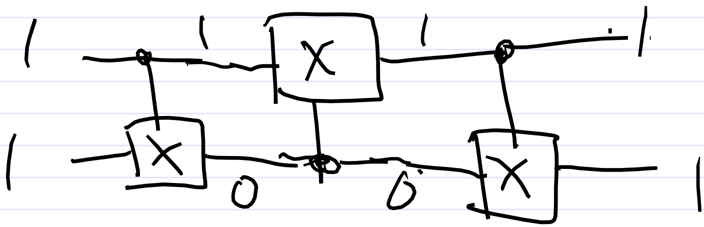
$$C_{01} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}$$

$|00\rangle$ $|10\rangle$

$|01\rangle \leftrightarrow |11\rangle$

$C_{ij} =$ 


$S_{ij} = C_{ij} \, C_{ji} \, C_{ij}$



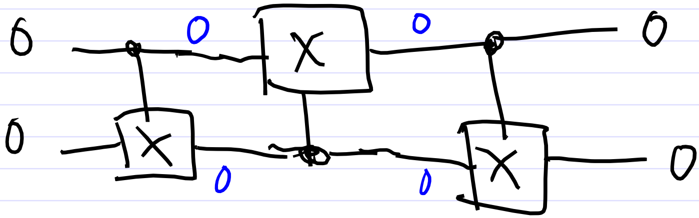

$|00\rangle \to |00\rangle$

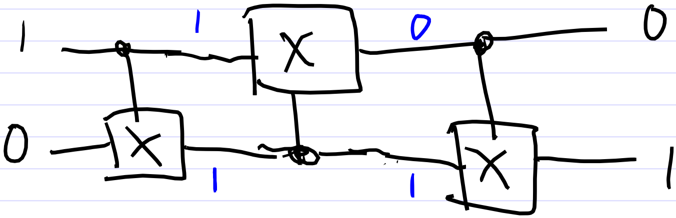
$|01\rangle \to |10\rangle$

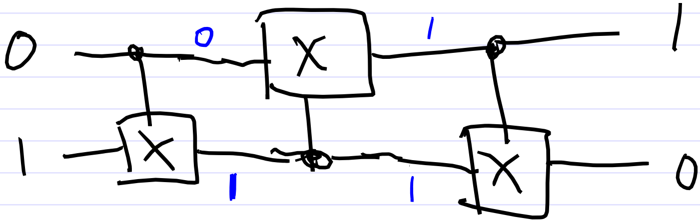
$|10\rangle \to |01\rangle$

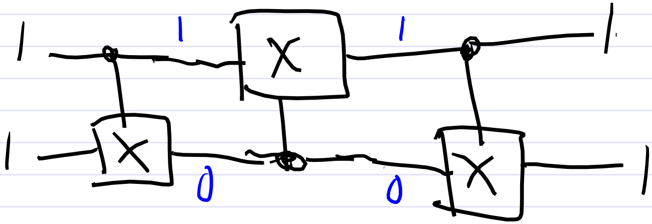
$|11\rangle \to |11\rangle$

<u>Swap</u>


0
0
X
0
0
X
X
0
0
0
0


1
1
X
0
0
0
X
1
1
X
1


0
0
X
1
1
1
X
1
1
X
0


1
1
X
1
1
1
X
0
0
X
1

More operators:

$Z \quad Z|0\rangle = |0\rangle \quad Z|1\rangle = -|1\rangle$

$$Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\boxed{\begin{aligned} ZX &= -XZ \\ X^2 &= 1 \\ Z^2 &= 1 \end{aligned}}$$

$$ZX|0\rangle = Z|1\rangle = -|1\rangle = -XZ|0\rangle$$

$$ZX|1\rangle = Z|0\rangle = |0\rangle = -XZ|1\rangle$$

Introduce "Hadamard" Op

$$H = \frac{1}{\sqrt{2}}(X+Z)$$

$$H^2 = 1$$

$$\begin{aligned} HXH &= \frac{1}{\sqrt{2}}(X+Z)\,X\,\frac{1}{\sqrt{2}}(X+Z) \\ &= \frac{1}{2}(X^3 + X^2 Z + ZXZ + ZX^2) \\ &= \frac{1}{2}(X + Z - X + Z) = Z \end{aligned}$$

$HXH = Z$

$HZH = H^2 X H^2 = X$

H exchanges $X \longleftrightarrow Z$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\left.\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ H|1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{aligned}\right\}$$