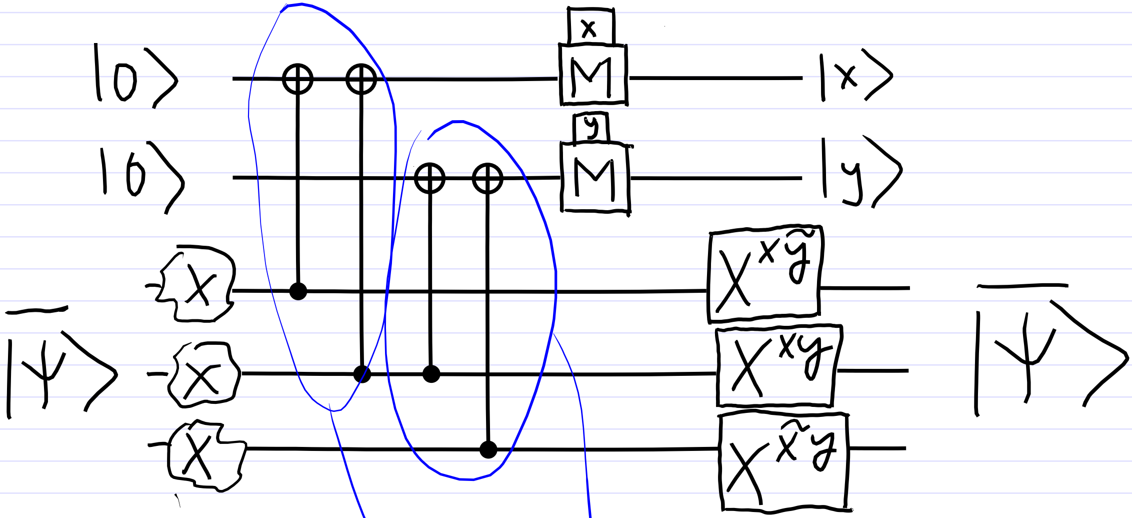
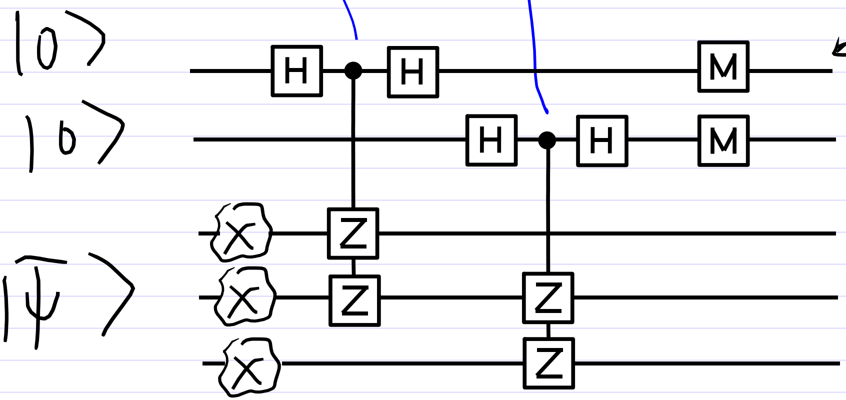
Lecture 21, 12 Nov 2020

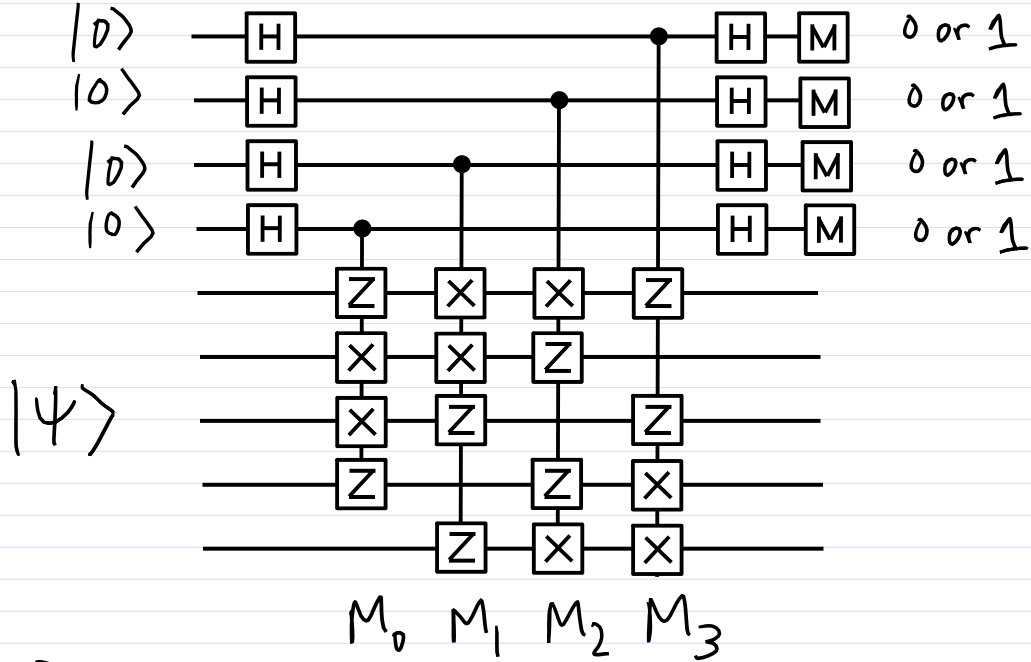


Use $X = HZH$



"measures" $Z_1 Z_2$

measures $Z_0 Z_1$



5-qubit code, encoding circuit.

To initialize state to $|\bar{0}\rangle$:

measure M_i's, projects to $\alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$ space.

measure $\bar{Z}$, gives $|\bar{0}\rangle$ or $|\bar{1}\rangle$

if $|\bar{1}\rangle$, apply $\bar{X}|\bar{1}\rangle = |\bar{0}\rangle$

How to implement arbitrary U?

- Universal gate set

  CNOT, Z, H, T $\quad$ $T^4 = Z$ ($\pi/8$ gate)

  gives any U

difficulty with 5-qubit code:

"hard" to get H, CNOT

instead use:

7-qubit "Steane code"

7-qubit code

$$M_0 = X_0 X_4 X_5 X_6 \qquad N_0 = Z_0 Z_4 Z_5 Z_6$$
$$M_1 = X_1 X_3 X_5 X_6 \qquad N_1 = Z_1 Z_3 Z_5 Z_6$$
$$M_2 = X_2 X_3 X_4 X_6 \qquad N_2 = Z_2 Z_3 Z_4 Z_6$$

$$M_i^2 = 1 = N_i^2 \qquad [M_i, M_j] = [N_i, N_j] = 0$$
$$[M_i, N_j] = 0$$

$$|\bar{0}\rangle = \frac{1}{2^{3/2}}(1+M_0)(1+M_1)(1+M_2)|0^7\rangle$$

$$|\bar{1}\rangle = \frac{1}{2^{3/2}}\prod_{i=0}^{2}(1+M_i)|1^7\rangle$$

$$\langle\bar{0}|\bar{0}\rangle = \frac{1}{2^3}2^3\langle 0^7|\prod_i(1+M_i)|0^7\rangle$$

$(1+M_i)^2 = 2(1+M_i)$

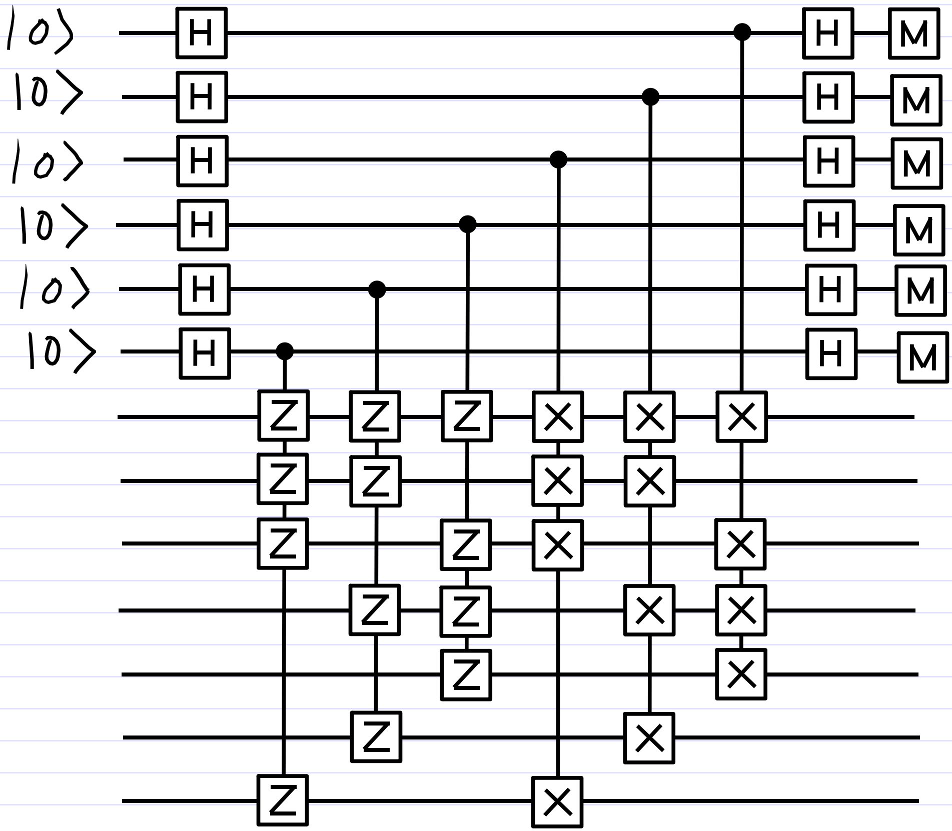
(only contribution) same for:

$$\langle\bar{1}|\bar{1}\rangle = 1$$

$$\langle\bar{0}|\bar{1}\rangle = \langle\bar{1}|\bar{0}\rangle = 0$$

(odd/even)

7-qubit measurement gates



N_0 N_1 N_2 M_0 M_1 M_2

$\prod (1+M_i)|0^7\rangle$

each term $|0001111\rangle$

has even # of 1s

$\langle 1^7|$ odd # of 1s so

always 0.

encoded $|\psi\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$

21 possible errors

$X_i \quad Y_i \quad Z_i$

+ 1 uncorrupted = 22

orthogonal 2d spaces

	1	X_0	X_1	X_2	X_3	X_4	X_5	X_6
M_0		•				•	•	•
M_1			•		•		•	•
M_2				•	•	•		•

	1	Z_0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
N_0		•				•	•	•
N_1			•		•		•	•
N_2				•	•	•		•

error syndromes:

X_i error $\Rightarrow$ look at which N_i's flip sign

Z_i error $\Rightarrow$ look at which M_i's flip sign

Y_i error $\Rightarrow$ look at pattern of both M, N flips [have to be SAME]

$$M_j X_i |\psi\rangle = X_i |\psi\rangle$$

$N_0 = -1$ error? X_0

$M_1 = M_2 = -1$ Z_3

all $M_0 = \dots = N_2 = -1$ Y_6

6

$$\bar{X} = X_0 X_1 \ldots X_6$$

$$\bar{Z} = Z_0 Z_1 \ldots Z_6$$

$$\bar{X}|\bar{0}\rangle = \bar{1} \qquad \bar{X}|\bar{1}\rangle = |\bar{0}\rangle$$

$$\bar{Z}|\bar{0}\rangle = |\bar{0}\rangle \qquad \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle$$

$$\bar{X}^2 = \bar{Z}^2 = 1 \qquad \bar{X}\bar{Z} = -\bar{Z}\bar{X}$$

Now $\bar{H} = H_0 H_1 H_2 H_3 H_4 H_5 H_6 = H^{\otimes 7}$

need to show:

$$\bar{H}|\bar{0}\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\rangle + |\bar{1}\rangle\right) \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\bar{H}|\bar{1}\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\rangle - |\bar{1}\rangle\right)$$

$$\langle\bar{0}|\bar{H}|\bar{0}\rangle = \langle\bar{0}|\bar{H}|\bar{1}\rangle = \langle\bar{1}|\bar{H}|\bar{0}\rangle$$

$$= -\langle\bar{1}|\bar{H}|\bar{1}\rangle = 1/\sqrt{2}$$

$$\langle \bar{x} | \bar{H} | \bar{y} \rangle = \qquad\qquad \bar{H} M_i = N_i \bar{H}$$

$$\frac{1}{2^3} \langle 0^7 | \bar{X}^x \prod_i (1+M_i) \; \bar{H} \; \prod_i (1+M_i) \bar{X}^y | 0^7 \rangle$$

$$= \frac{1}{2^3} \langle 0^7 | \bar{X}^x \prod_i (1+N_i) \; \bar{H} \; \prod_i (1+N_i) \bar{X}^y | 0^7 \rangle$$

$$= 2^3 \langle 0^7 | \bar{X}^x \bar{H} \bar{X}^y | 0^7 \rangle$$

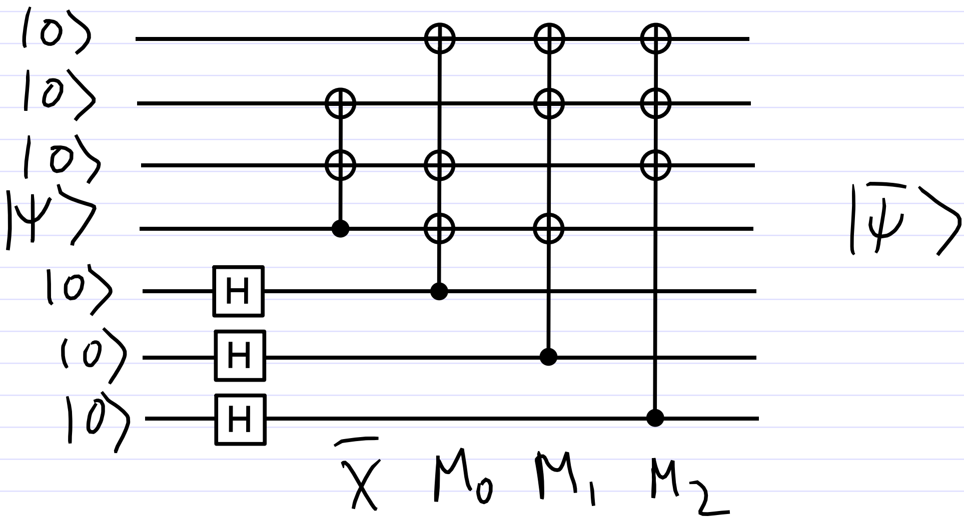
$$= 2^3 \left(\langle 0 | X^x H X^y | 0 \rangle \right)^7$$

$$= \cancel{2^3 \left(\langle 0 | X^x H X^y | 0 \rangle \right)^6} \langle 0 | X^x H X^y | 0 \rangle$$

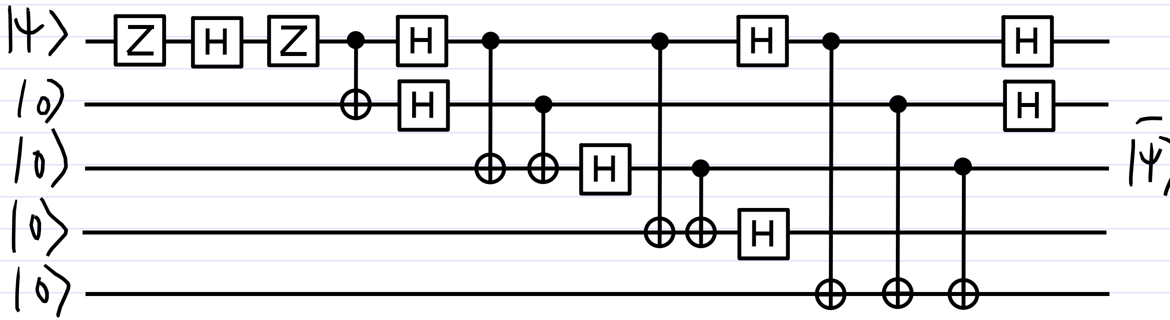
$$= \langle 0 | X^x H X^y | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

✓

7-qubit encoding circuit



Compare to 5 qubit encoding circuit



7-qubit CNOT

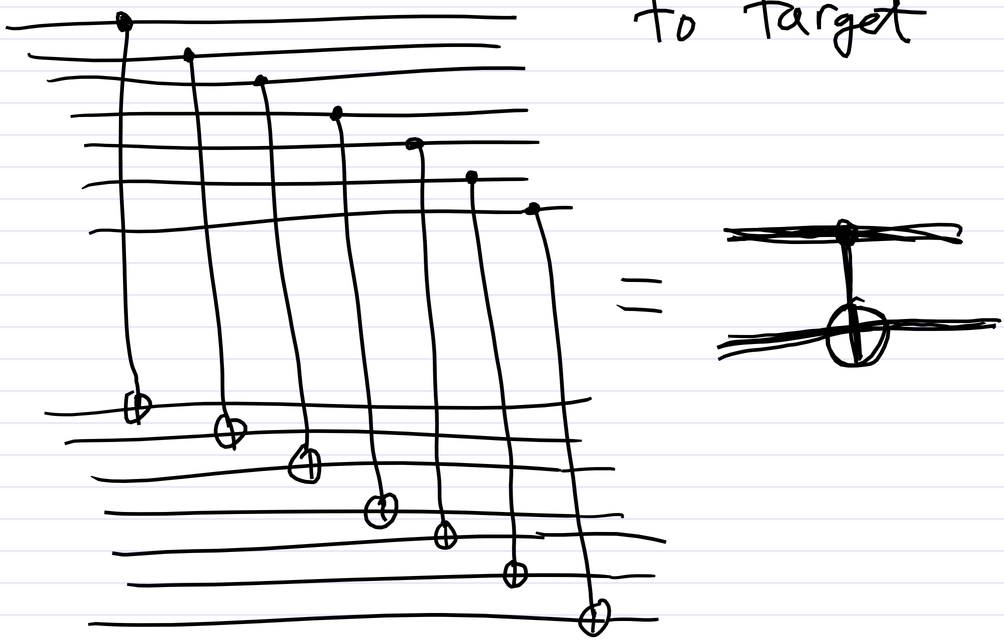
also simple structure:

$|\bar{0}\rangle = \prod_i (1+M_i)|0^7\rangle$

$|\bar{1}\rangle = \bar{X}|\bar{0}\rangle$

If control is $|\bar{0}\rangle$, then pattern of M_i's applies $\prod_i (1+M_i)$ to target, no effect.

If control is $|\bar{1}\rangle$, then applies additional $\bar{X}$ to target



and $\overline{\text{CNOT}} = \text{CNOT}^{\otimes 7}$

$\overline{\text{CNOT}}$, $\bar{H}$, $\bar{X}$, $\bar{Z}$ all parallelize, So 7-qubit code can be made fault tolerant,

BUT current qubits still not stable enough —

Need surface code, thousands of physical qubits per logical qubit.

Then can preserve single qubit phase coherence for millions of years