Lecture 17, 29 Oct 2020

Grover's Algorithm

Search.

Given N items, 1 "marked"

look at K of them

prob $= K/N$ of finding it.

Quantum: "look" at $\sim\sqrt{N}$

e.g. Database

or $p = m^2 + n^2$ $\quad 4k+1$

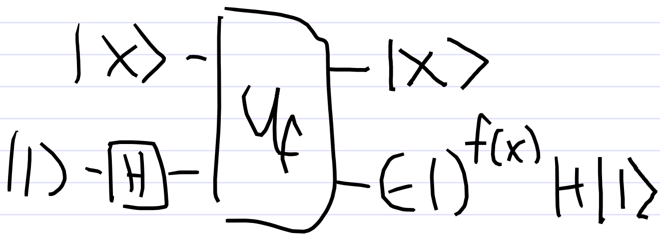
$\sim\sqrt{p/2}$, $\sqrt{p}$

$$f(x) = \begin{cases} 0 & x \neq a \\ 1 & x = a \end{cases}$$ marked item

$N = 2^n$

$$U_f |x\rangle_n |y\rangle_1 = |x\rangle |y \oplus f(x)\rangle_1$$

"phase kickback"



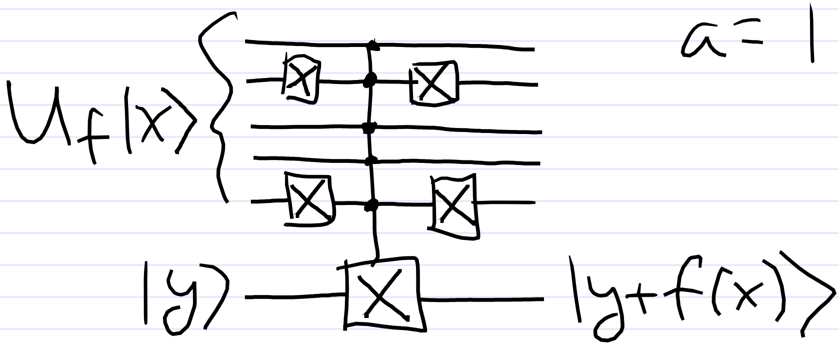
$$U_f(|x\rangle \otimes \underbrace{H|1\rangle}) = (-1)^{f(x)} (|x\rangle \otimes \underbrace{H|1\rangle})$$

$$V(x)|x\rangle_n = (-1)^{f(x)} |x\rangle_n = \begin{cases} |x\rangle & x \neq a \\ -|a\rangle & x = a \end{cases}$$

$$V|\psi\rangle = |\psi\rangle - 2|a\rangle\langle a|\psi\rangle$$

$$V_f = \underline{1} - 2|a\rangle\langle a|$$

embodies f



$a = 10110$

$$|\varphi\rangle = H^{\otimes n}|0\rangle = \frac{1}{2^{n/2}} \sum_{0 \le x < 2^n} |x\rangle$$

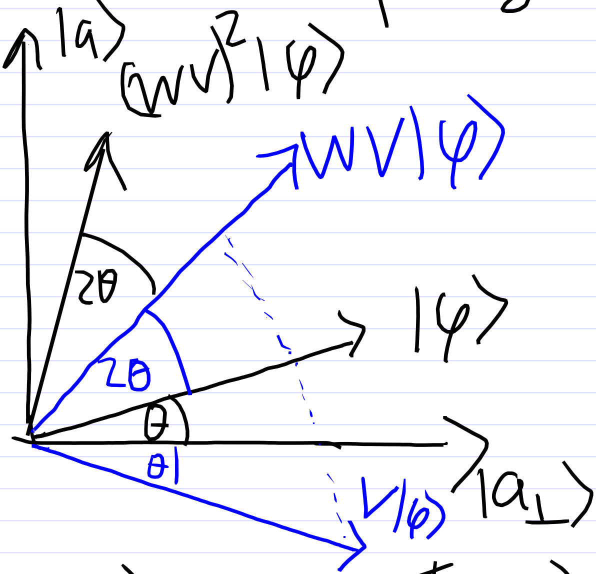
also need

$$W = 2|\varphi\rangle\langle\varphi| - 1$$

inverts states orthogonal to $|\varphi\rangle$

$$V = \underline{1} - 2|a\rangle\langle a|$$

work in 2d space generated by $|a\rangle, |\varphi\rangle$



$$\langle a|\varphi\rangle = \cos\left(\pi/2 - \theta\right) = \sin\theta = \frac{1}{2^{n/2}}$$

$$\approx \theta \approx \frac{1}{\sqrt{N}}$$

WV is a rotation (by what angle?)
by 2θ

(or $V = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\quad W = R_\theta V R_{-\theta}$ $\quad R_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$WV = R_\theta V R_{-\theta} V = R_{2\theta}$)

Apply $(WV)^{\ell}$ $\quad \theta \approx \frac{1}{2^{n/2}}$

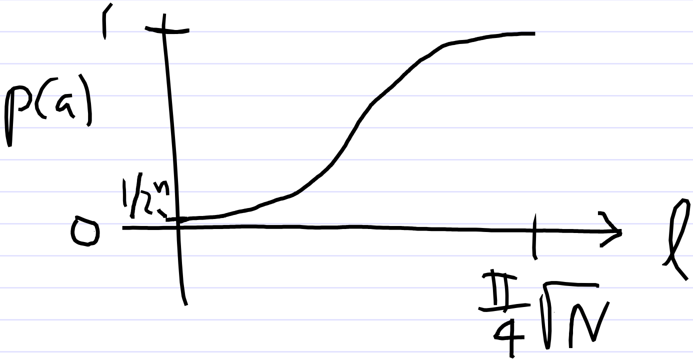
with $(2\ell+1)\frac{1}{2^{n/2}} = \pi/2$

gives close as possible to $|a\rangle$

$$\ell \approx \frac{\pi}{4} 2^{n/2} = \pi/4 \sqrt{N}$$

$$p(a) = |\langle a|(WV)^{\ell}|\varphi\rangle|^2$$

$$= \sin^2 (2\ell+1)\theta = \sin^2\left(\frac{2\ell+1}{2^{n/2}}\right)$$



m marked states

$$f(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$$

$$V_k|x\rangle = (-1)^{f(x)}|x\rangle$$

$Y =$ set of marked states

$$|Y| = m$$

$$|\varphi\rangle = \frac{1}{2^{n/2}} \sum_{0 \le x < 2^n} |x\rangle = \cos\theta|no\rangle + \sin\theta|yes\rangle$$

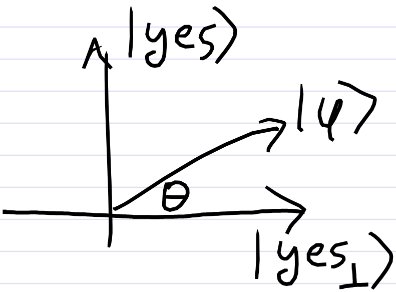
$$|yes\rangle = \frac{1}{\sqrt{m}} \sum_{x|f(x)=1} |x\rangle \qquad |no\rangle = \frac{1}{\sqrt{2^n - m}} \sum_{x|f(x)=0} |x\rangle$$

$$\sin\theta = \langle yes|\varphi\rangle = \sqrt{\frac{m}{2^n}}$$

$$\cos\theta = \langle no|\varphi\rangle = \sqrt{1 - m/2^n}$$

$$V = 1 - 2|\text{yes}\rangle\langle\text{yes}|$$

$$W = 2|\varphi\rangle\langle\varphi| - 1$$



WV rotates by 2θ

$$(2\ell+1)\theta \approx \pi/2$$

$$\theta \approx \sqrt{m/2^n} = \sqrt{m/N}$$

$$\ell \approx \pi/4 \sqrt{N/m}$$

$\langle yes|(WV)^\ell|\varphi\rangle$ is periodic

$\approx \sin 2\ell\theta$ in $\ell \pm \pi/\theta$

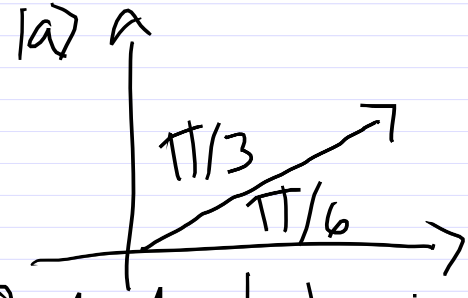
so period $= \dfrac{\pi 2^{n/2}}{\sqrt{m}}$.

Run QFT to get period, gives m

then Grover $\frac{\pi}{4}\sqrt{N/m}$ times

A special case: $m=1$ $n=2$ $(N=4)$

$\sin\theta = \langle a|\varphi\rangle = \frac{1}{2^{n/2}} = 1/2$ $\quad\theta = \pi/6$



WV rotates by $\pi/3$

single WV exactly $|a\rangle$

QM: 1 | Classically expect: $\frac{1}{4}\cdot 1 + \frac{3}{4}\frac{1}{3}2 + \frac{1}{2}3$

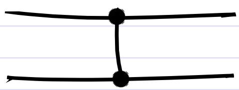
$= 2\,{}^1/_4$

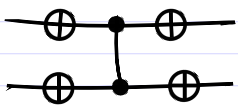
How to implement W

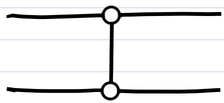
- $W = 1 - 2|\psi\rangle\langle\psi|$

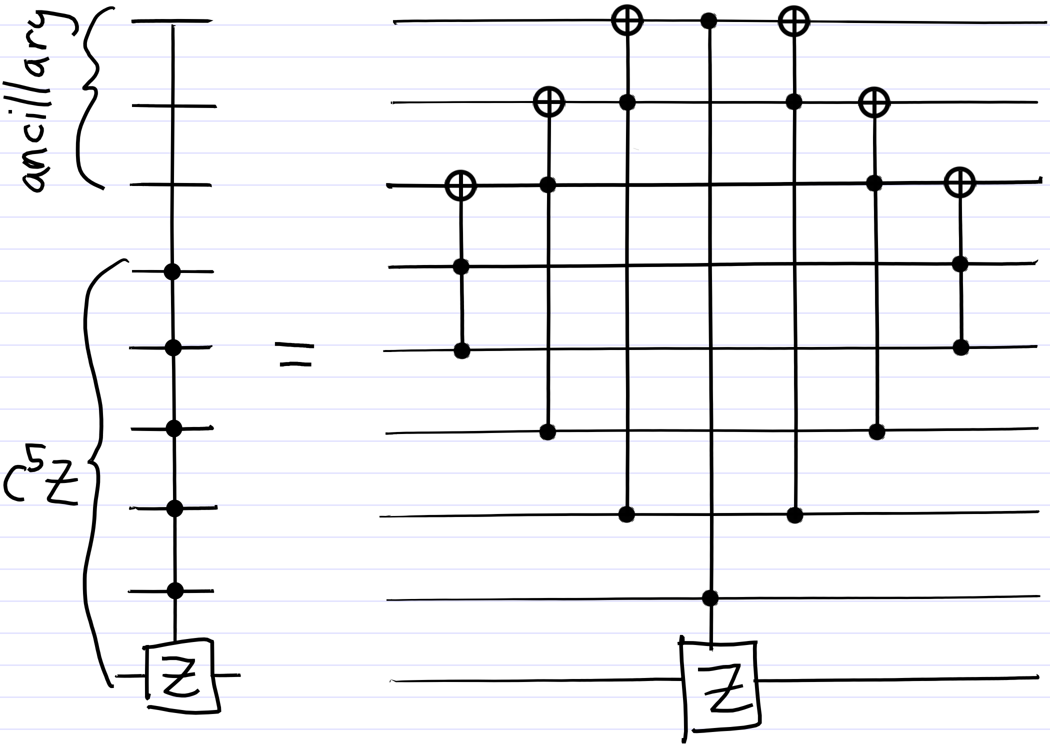
$$= H^{\otimes n}\underbrace{\left(1 - 2|0\rangle_n {}_n\langle 0|\right)}_{X^{\otimes n} C^{n-1}Z\, X^{\otimes n}} H^{\otimes n}$$

$n=2$

$1 - 2|11\rangle\langle 11| =$ 

$1 - 2|00\rangle\langle 00| =$ 

$=$ 



One way to construct multiple $C^{n-1}Z$ and W