Lecture 13, 15 Oct 2020

what is the probability that 15 Oct is a Thu?

$365 = 1 \bmod 7$

$400 + 100 - 4 + 1 = 497$

$= 0 \bmod 7$!

57 58 56 58 56 58 57

$A = a \bmod N \quad B = b \bmod N$

$(A+B) \bmod N = (a+b) \bmod N$

$A \cdot B \bmod N = (a+mN)(b+nN)$

$= ab + \cancel{anN} + \cancel{bmN} + \cancel{mnN}$

QC: $f(x) = b^x \bmod N$

$$U_f H^{\otimes n} |0\rangle_n |0\rangle_{n_0} = \frac{1}{2^{n/2}} \sum_{0 \le x < 2^n} |x\rangle_n |f(x)\rangle_{n_0}$$

measure output, input will be

$$|\psi\rangle_n = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle |f(x_0)\rangle$$

Challenge: get rid of x_0 to learn about r.

Solution: $H^{\otimes n} \longrightarrow U_{FT}$

$$U_{FT}|x\rangle = \frac{1}{2^{n/2}} \sum_{0 \le y < 2^n} e^{2\pi i x y / 2^n} |y\rangle_n$$

Ordinary F.T. $f(x) = \int_w e^{iwx} \underline{\hat{f}_w}$

$$\tilde{\hat{f}}_w = \int_{x'} e^{-iwx'} f(x')$$

$$\underline{\int_x e^{i(w-w')x'} = \delta(w-w')}$$

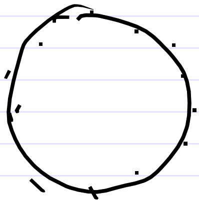
verify $\left| U_{FT}|x\rangle \right| = 1$

$$\langle x'|U_{FT}^{\dagger} U_{FT}|x\rangle \qquad \langle x'|x\rangle = \delta_{x'x}$$

$$= \frac{1}{2^n}\sum_{y,y'} \langle y'|\, e^{-2\pi i x'y/2^n}\, e^{2\pi i x y/2^n} |y\rangle$$

$$= \frac{1}{2^n}\sum_{0\le y<2^n} e^{2\pi i y(x-x')/2^n}$$

$$= \delta_{x,x'}$$



[sum of phases around unit circle in $\mathbb{C}$, or see next page, which uses

$$S(r) = 1 + r + r^2 + \dots r^{n-1}$$

$$rS(r) = S(r) + r^n - 1$$

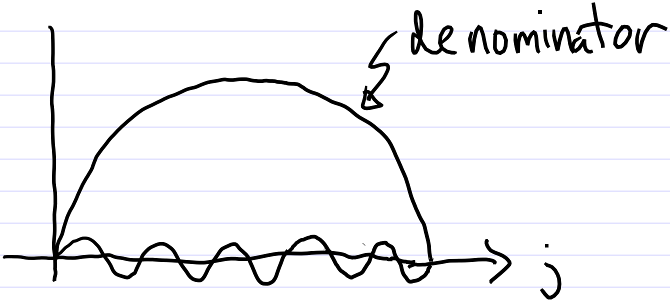
$$(1-r)S(r) = 1 - r^n \Rightarrow S(r) = \frac{1-r^n}{1-r}$$
]

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i jk/N} = \delta_{j0}$$

Let $\omega = e^{2\pi i/N}$

$$\frac{1}{N}\sum_{k=0}^{N-1} = \frac{1}{N}\,\frac{1-\omega^{Nj}}{1-\omega^{j}} = \frac{1}{N}\,\frac{\omega^{Nj/2}}{\omega^{j/2}}\,\frac{\omega^{Nj/2}-\omega^{-Nj/2}}{\omega^{j/2}-\omega^{-j/2}}$$

$$* \qquad = \frac{1}{N}\,\omega^{(N-1)j/2}\,\frac{\sin \pi j}{\sin \pi j/N}$$



Numerator: N half periods

Vanishes for $j \neq 0$

$$j \to 0 \longrightarrow \frac{1}{N}\,\frac{\pi}{\pi/N} = 1$$

$$= \delta_{j0}$$

(*recall $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$)

Effect on amplitudes

$$U_{FT}\left(\sum_{0\le x<2^n} \gamma(x)|x\rangle\right) = \sum_{0\le x<2^n} \tilde{\gamma}(x)|x\rangle$$

$$\tilde{\gamma}(x) = \frac{1}{2^{n/2}} \sum_{0\le y<2^n} e^{2\pi i xy/2^n}\gamma(y)$$

(usual active vs. passive)

$\gamma(x)$ are 2^n complex numbers

$\tilde{\gamma}(x)$ " "

ordinary FT $\sim (2^n)^2$

FFT $\sim n(2^n)$ $\quad \left(\begin{array}{c} N\ln N \\ N=2^n \end{array}\right)$

Q.C. $O(n^2)$

$$|\psi\rangle_n = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle |f(x_0)\rangle$$

$$U_{FT}|\psi\rangle_n = \frac{1}{2^{n/2}} \sum_y \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (x_0+kr)y/2^n} |y\rangle$$

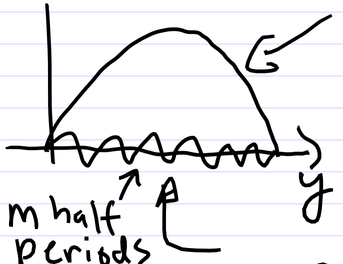
$$= \sum_{0<y\leq 2^n} \underbrace{e^{2\pi i x_0 y}}_{\text{(just a phase)}} \frac{1}{\sqrt{2^n m}} \left(\sum_{k=0}^{m-1} e^{2\pi i k r y/2^n} \right) |y\rangle$$

now measure y, x_0 disappears:

$$p(y) = |\quad|^2 = \frac{1}{2^n m} \left| \sum_{k=0}^{m-1} e^{2\pi i k r y/2^n} \right|^2$$

Peaked at $y \approx 2^n/r \; j$

$$p(y) = \frac{1}{2^n m}\left|\frac{1-e^{2\pi i m r y/2^n}}{1-e^{2\pi i r y/2^n}}\right|^2$$



$$= \frac{1}{2^n m}\,\frac{\sin^2 \pi m r y/2^n}{\sin^2 \pi r y/2^n}$$

Now only appreciable near $y = j \cdot 2^n/r$

needs classical work to extract r from (nearest integer to) $j \cdot 2^n/r$

Next time:

1) get r from nearest integer y to some $j \cdot 2^n / r$

2) Show that QFT is $O(n^2)$