ECE4070 Homework #3 Solutions (Farhan Rana)

3.1

a) $\vec{a}_1 = a\left(\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}\right)$

$\vec{a}_2 = a\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)$

b) $\Omega_2 = |\vec{a}_1 \times \vec{a}_2| = \frac{\sqrt{3}}{2}a^2$

c) See attached figure.

d) $\vec{b}_1 = 2\pi \frac{\vec{a}_2 \times \hat{z}}{\Omega_2} = \frac{4\pi}{\sqrt{3}a}\left(-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\right)$

$\vec{b}_2 = 2\pi \frac{\hat{z} \times \vec{a}_1}{\Omega_2} = \frac{4\pi}{\sqrt{3}a}\left(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right)$

e) $\Pi_2 = |\vec{b}_1 \times \vec{b}_2| = \frac{(2\pi)^2}{\Omega_2} = \frac{4\pi^2}{\frac{\sqrt{3}}{2}a^2} = \frac{8\pi^2}{\sqrt{3}a^2}$

f) see attached plot. lattice is hexagonal.

g) See attached plot

3.2

a) For FCC lattice: $\vec{a}_1 = \frac{a}{2}(\hat{y}+\hat{z})$ $\vec{a}_2 = \frac{a}{2}(\hat{x}+\hat{z})$ $\vec{a}_3 = \frac{a}{2}(\hat{x}+\hat{y})$

$\Omega_3 = |\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)| = \frac{a^3}{4}$

$\vec{b}_1 = \frac{2\pi}{\Omega_3}(\vec{a}_2 \times \vec{a}_3) = \frac{2\pi}{a}(\hat{z}+\hat{y}-\hat{x})$ $\vec{b}_2 = \frac{2\pi}{\Omega_3}(\vec{a}_3 \times \vec{a}_1) = \frac{2\pi}{a}(\hat{x}-\hat{y}+\hat{z})$

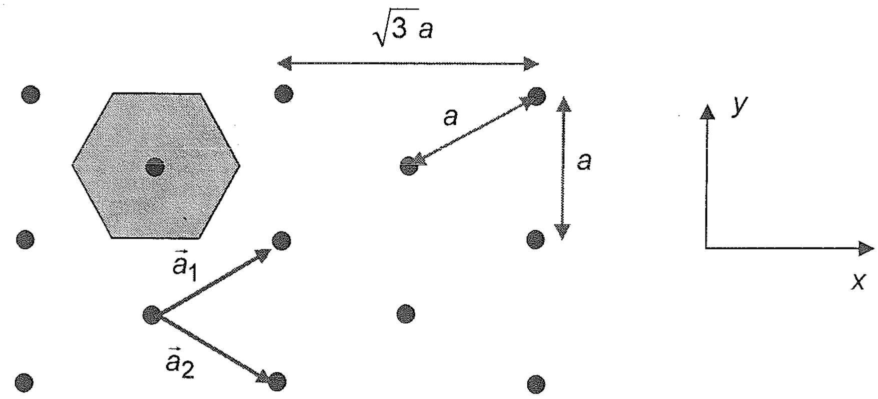
$\vec{b}_3 = \frac{2\pi}{a}(\hat{x}+\hat{y}-\hat{z})$ $\Rightarrow$ $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ correspond to a BCC lattice with a unit cell dimension of $\frac{4\pi}{a}$.

b) For BCC lattice: $\vec{a}_1 = \frac{a}{2}(-\hat{x}+\hat{y}+\hat{z})$ $\vec{a}_2 = \frac{a}{2}(\hat{x}-\hat{y}+\hat{z})$

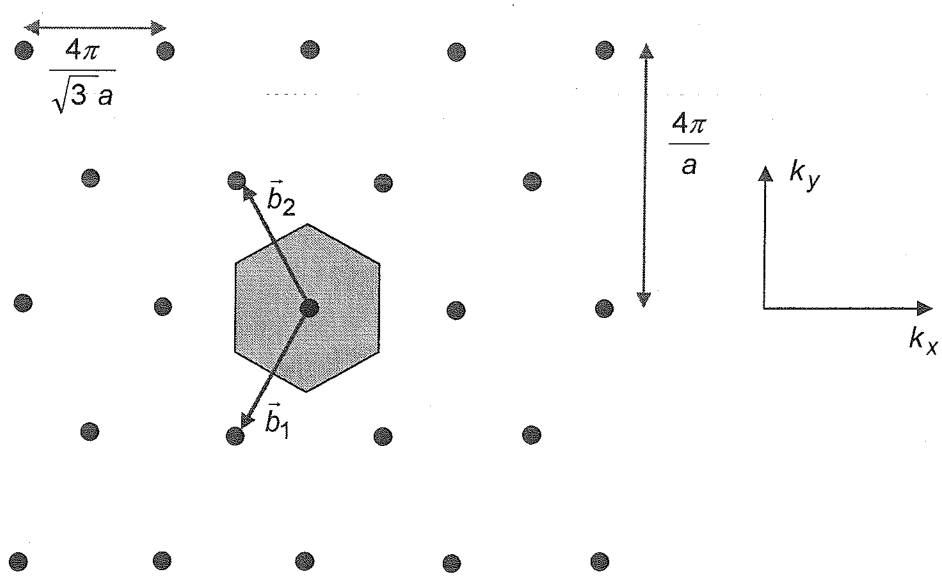
$\vec{a}_3 = \frac{a}{2}(\hat{x}+\hat{y}-\hat{z})$

Problem 3.1 plots

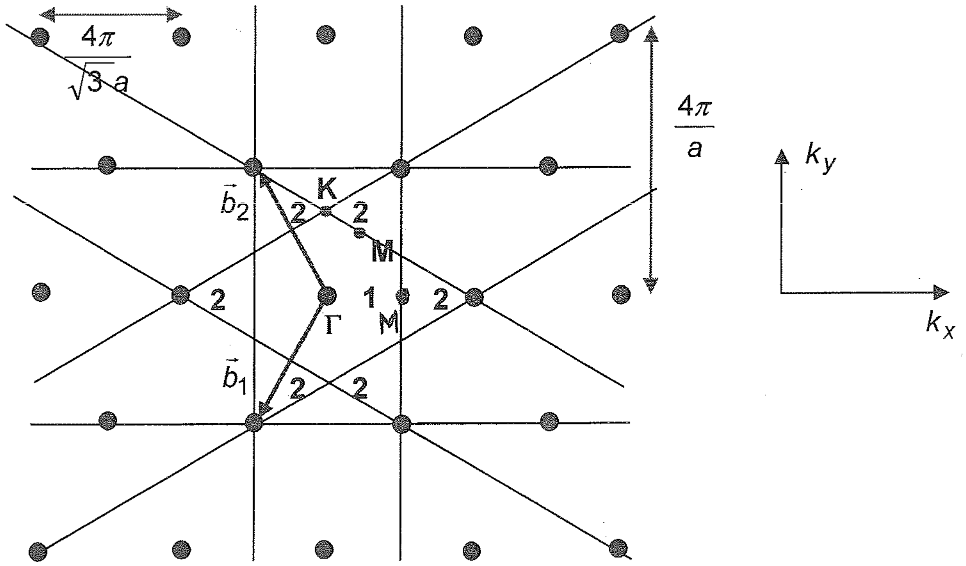
Direct Lattice:



Reciprocal Lattice:



Bragg Planes and Higher BZs



$\Omega_3 = |\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)| = \frac{a^3}{2}$

$\vec{b}_1 = \frac{2\pi}{a}(\hat{y}+\hat{z}) \quad \vec{b}_2 = \frac{2\pi}{a}(\hat{x}+\hat{z}) \quad \vec{b}_3 = \frac{2\pi}{a}(\hat{x}+\hat{y})$

$\Rightarrow$ $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ correspond to a fcc lattice with unit cell size equal to $\frac{4\pi}{a}$.

3.3

a) $\vec{a}_1 = \sqrt{3}a\,\hat{x} \quad \hat{a}_2 = a\,\hat{y}$

b) $\Omega_2 = |\vec{a}_1 \times \vec{a}_2| = \sqrt{3}a^2$.

c) See attached. There are two atoms per primitive cell: one black and one red.

d) $\vec{b}_1 = \frac{2\pi}{\sqrt{3}a}\hat{x} \quad \vec{b}_2 = \frac{2\pi}{a}\hat{y}$.

e) $\Pi_2 = |\vec{b}_1 \times \vec{b}_2| = \frac{(2\pi)^2}{\Omega_2}$.

f) See attached

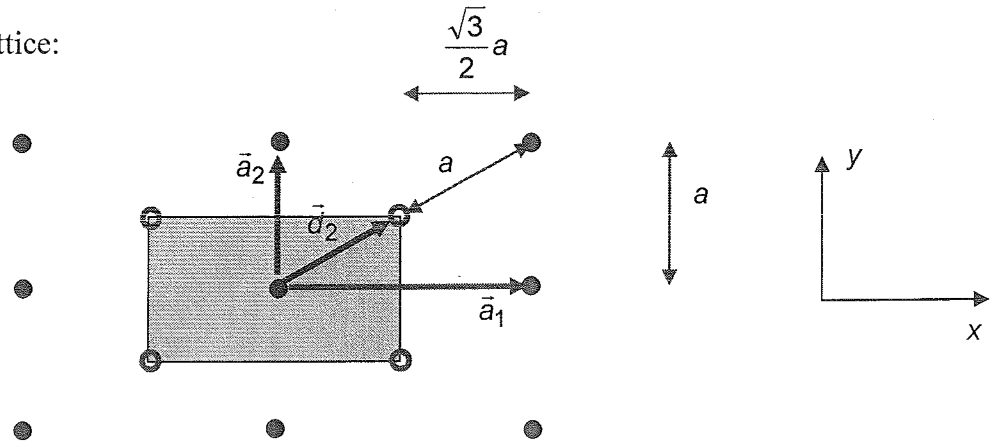
3.4

a) See attached.

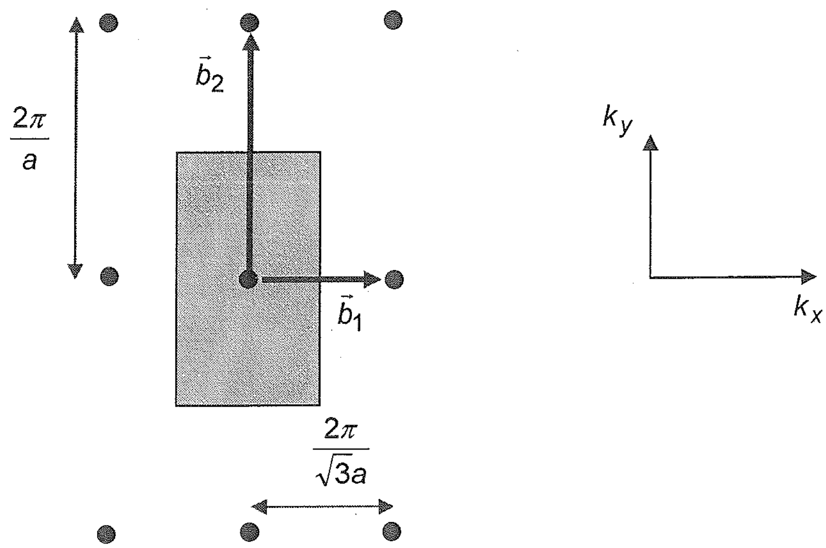
b) See attached.

Problem 3.3 (f)

Direct lattice:

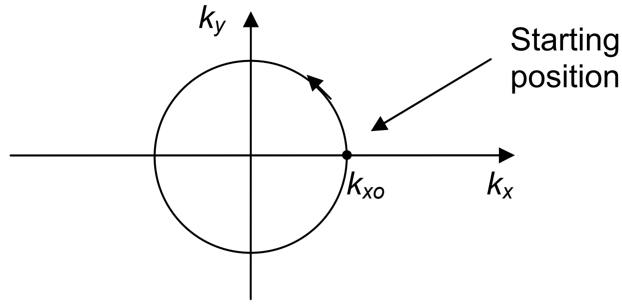


Reciprocal lattice:

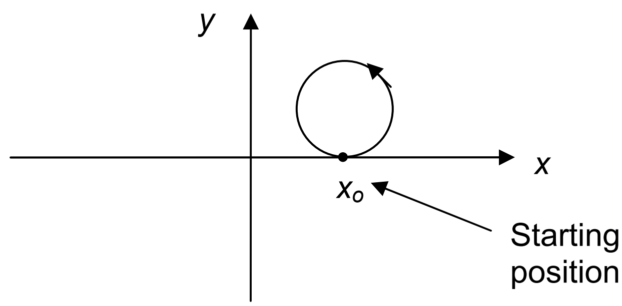


3.4

a) $\dfrac{d\,\hbar\vec{k}}{dt} = -e\vec{v}(\vec{k})\times\vec{B} \quad\Rightarrow\quad \dfrac{d\,\vec{k}}{dt} = -\dfrac{e}{m}(\vec{k}\times\vec{B})$



b) $\dfrac{d\,\hbar\vec{k}}{dt} = -e\vec{v}(\vec{k})\times\vec{B} \quad\Rightarrow\quad \dfrac{d^2\,\vec{r}}{dt^2} = \dfrac{d\,\vec{v}}{dt} = -\dfrac{e}{m}(\vec{v}\times\vec{B})$



c) The frequency of the periodic motion is as found in homework 1 and equals,

$$\omega_c = \frac{eB_o}{m}$$

So the time period is,

$$T = \frac{2\pi}{\omega_c}$$

d) Start from,

$$\frac{d\,\hbar\vec{k}(t)}{dt} = -e\vec{v}(\vec{k})\times\vec{B}$$

Take the dot product on both sides with $\vec{k}$ and note that the RHS becomes zero,

$$\vec{k}\cdot\frac{d\,\hbar\vec{k}(t)}{dt} = -e\,\vec{k}\,.\,\left(\vec{v}(\vec{k})\times\vec{B}\right) = -\frac{e\hbar}{m}\vec{k}\,.\,(\vec{k}\times\vec{B}) = 0$$

$$\Rightarrow \frac{d\,(\hbar\,\vec{k}\cdot\vec{k})}{dt} = 0 \quad\Rightarrow\quad \frac{d\,(\hbar^2\,\vec{k}\cdot\vec{k}/2m)}{dt} = 0 \quad\Rightarrow\quad \frac{d\,E(\vec{k})}{dt} = 0$$

e) In the presence of the magnetic field the entire distribution of filled electron states in k-space rotates as indicated in the answer to part (a). However, the distribution remains completely spherically symmetric and therefore the net current given by the expression below would equal zero just as was the case in the absence of the magnetic field,

$$\vec{J} = -2\,e\times\int\frac{d^2\vec{k}}{(2\pi)^2}\,f(\vec{k})\,\vec{v}(\vec{k}) = 0$$