ECE 407 Homework #1 Solutions (Farhan Rana).

1.1 (Ignore the angled brackets below)

a) $\vec{E}=0 \Rightarrow m\dfrac{d\langle v_x\rangle}{dt} = -eB_o\langle v_y\rangle$ and $m\dfrac{d\langle v_y\rangle}{dt} = eB_o\langle v_x\rangle$.

$\Rightarrow \dfrac{d^2\langle v_x\rangle}{dt^2} = -\omega_c^2\langle v_x\rangle$ and $\dfrac{d^2\langle v_y\rangle}{dt^2} = -\omega_c^2\langle v_y\rangle$ $\quad \omega_c = \dfrac{eB_o}{m}$.

$\Rightarrow \langle v_x\rangle = A\cos\omega_c t \quad \& \quad \langle v_y\rangle = A\sin\omega_c t$

b) $\dfrac{d\langle u_x\rangle}{dt} = A\cos\omega_c t \Rightarrow \langle u_x\rangle = \dfrac{A}{\omega_c}\sin\omega_c t$

and $\dfrac{d\langle u_y\rangle}{dt} = A\sin\omega_c t \Rightarrow \langle u_y\rangle = -\dfrac{A}{\omega_c}\cos\omega_c t + \dfrac{A}{\omega_c}$.

Position vector of the electron is $= \vec{r}(t) = \dfrac{A}{\omega_c}\hat{y} + \dfrac{A}{\omega_c}\left[\sin\omega_c t\,\hat{x} - \cos\omega_c t\,\hat{y}\right]$.

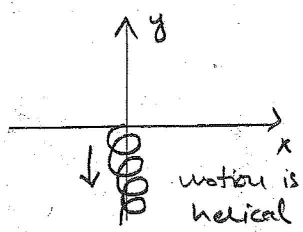
$\Rightarrow$ circular motion centered at $\dfrac{A}{\omega_c}\hat{y}$ and of radius $\dfrac{A}{\omega_c}$.

c) $m\dfrac{d\langle v_x\rangle}{dt} = -eE_x - eB_o\langle v_y\rangle$ $\quad$ and $m\dfrac{d\langle v_y\rangle}{dt} = eB_o\langle v_x\rangle$.

$\dfrac{d^2\langle v_y\rangle}{dt^2} = -\omega_c^2\langle v_y\rangle - \omega_c\dfrac{e}{m}E_x$. $\quad \& \quad \dfrac{d^2\langle v_x\rangle}{dt^2} = -\omega_c^2\langle v_x\rangle$.

$\Rightarrow \langle v_x\rangle = A\cos\omega_c t \qquad \langle v_y\rangle = A\sin\omega_c t - \dfrac{E_x}{B_o}$

$\Rightarrow$ There is a constant velocity component in the $-\hat{y}$ direction!



d) $m\dfrac{d\langle v_x\rangle}{dt} = -eE_x - eB_o\langle v_y\rangle - \dfrac{m\langle v_x\rangle}{\tau} = 0$.

$m\dfrac{d\langle v_y\rangle}{dt} = -eE_y + eB_o\langle v_x\rangle - \dfrac{m\langle v_y\rangle}{\tau} = 0$.

$$\Rightarrow \langle v_x\rangle = \frac{\omega_c^2\dfrac{E_y}{B_o} - \dfrac{\mu}{\tau^2}E_x}{\omega_c^2 + 1/\tau^2} \qquad \mu = \frac{e\tau}{m}.$$

$$\langle v_y\rangle = \frac{-\omega_c^2\dfrac{E_x}{B_o} - \dfrac{\mu}{\tau^2}E_y}{\omega_c^2 + 1/\tau^2}$$

e) $J_x = n(-e)v_x$ and $J_y = n(-e)v_y$.

$$J_x = n\left\{\frac{-e\omega_c^2\frac{E_y}{B_0} + \frac{e\mu}{\tau^2}E_x}{\omega_c^2 + 1/\tau^2}\right\}$$

$$J_y = n\left\{\frac{e\omega_c^2\frac{E_x}{B_0} + \frac{e\mu}{\tau^2}E_y}{\omega_c^2 + 1/\tau^2}\right\}$$

f) $J_y = 0 \Rightarrow E_y = -\dfrac{(\omega_c\tau)^2\frac{E_x}{B_0}}{\mu} = B_0\langle v_x\rangle$.

g) $\rho_H = \dfrac{E_y}{J_x} = \dfrac{B_0\langle v_x\rangle}{n(-e)\langle v_x\rangle} = -\dfrac{B_0}{ne} \Rightarrow \rho_H$ is -ve for electrons.

1.2

a) $\sigma = \dfrac{1}{2.2\times10^{-8}}$ S/m. $\Rightarrow \sigma = \dfrac{ne^2\tau}{m} \Rightarrow n = 5.39\times10^{28}/m^3$.

$\Rightarrow \omega_p = \sqrt{\dfrac{ne^2}{\varepsilon_0 m}} = 1.3\times10^{16}$ rad/sec $= 2\times10^{15}$ Hz $\Rightarrow$ UV range.

b) $\omega_p = \sqrt{\dfrac{ne^2}{\varepsilon_0 m}} = 1.78\times10^{13}$ rad/sec $= 2.83\times10^{12}$ Hz $\Rightarrow$ IR range.

1.3

a) Set $\det\begin{bmatrix} E-\lambda & t \\ t & E-\lambda\end{bmatrix} = 0 \Rightarrow (E-\lambda)^2 = t^2 \Rightarrow \lambda = E\pm t$.

$\Rightarrow E_1 = E-t \qquad E_2 = E+t$.

b) Need to find eigenstates of $\begin{bmatrix} E & t \\ t & E\end{bmatrix}$ corresponding to the eigenvalues E_1 and E_2. These are: $\begin{bmatrix} \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2}\end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt2} \\ +\frac{1}{\sqrt2}\end{bmatrix}$.

$\Rightarrow |\psi_1\rangle = \frac{1}{\sqrt2}\left[|\phi_1\rangle - |\phi_2\rangle\right] \qquad |\psi_2\rangle = \frac{1}{\sqrt2}\left[|\phi_1\rangle + |\phi_2\rangle\right]$.

1.4

a) $f(t) = \sum_n \delta(t-nT)$.

$$f(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\sum_n \delta(t-nT) = \sum_n e^{i\omega nT} = \frac{2\pi}{T}\sum_n \delta\left(\omega - n\frac{2\pi}{T}\right).$$

b) $$g(\omega) = \int_{-W/2}^{W/2} dt\, e^{i\omega t} = W \frac{\sin\left(\omega \frac{W}{2}\right)}{\left(\frac{\omega W}{2}\right)}.$$

c) $$h(t) = f(t) \otimes g(t) \Rightarrow h(\omega) = f(\omega) \cdot g(\omega).$$

$$\Rightarrow h(\omega) = W \frac{\sin\left(\frac{\omega W}{2}\right)}{\left(\frac{\omega W}{2}\right)} \cdot \frac{2\pi}{T} \sum_n \delta\left(\omega - n\frac{2\pi}{T}\right).$$

$$= \frac{2\pi W}{T} \sum_n \delta\left(\omega - n\frac{2\pi}{T}\right) \frac{\sin\left(n\frac{\pi W}{T}\right)}{\left(n\frac{\pi W}{T}\right)}.$$

d) $$h(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, h(\omega) = \sum_n \frac{W}{T} \frac{\sin\left(n\frac{\pi W}{T}\right)}{\left(\frac{n\pi W}{T}\right)} e^{-i n \frac{2\pi}{T} t}.$$

e) $$h(\vec{k}) = \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}\, h(\vec{r}).$$

$$= L_x \frac{\sin\left(\frac{k_x L_x}{2}\right)}{\left(\frac{k_x L_x}{2}\right)} \cdot L_y \frac{\sin\left(\frac{k_y L_y}{2}\right)}{\left(\frac{k_y L_y}{2}\right)} \cdot L_z \cdot \frac{\sin\left(\frac{k_z L_z}{2}\right)}{\left(\frac{k_z L_z}{2}\right)}.$$