ECE 4070/MSE 5470
Physics of Semiconductors and Nanostructures
Exam 1, February 24, 2015

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Instructions:

- There are **THREE** problems in this exam
- Every problem must be done in the blue booklet
- Only work done on the blue exam booklets will be graded. Do not attach your own sheets to the exam booklets under any circumstances
- To get partial credit you must show all the relevant work
- Correct answers with wrong reasoning will not get points
- All questions do not carry equal points
- All questions do not have the same level of difficulty, use your time judiciously

DO NOT WRITE IN THIS SPACE

1 Quantum-Mechanics Recap [40 points]

We derived in class that the allowed wavefunctions representing an electron on a circular ring of circumference L is $\psi_n(x) = \frac{1}{\sqrt{L}} e^{ik_n x}$, where $k_n = \frac{2\pi}{L} n$ are *quantized* because $n = 0, \pm 1, \pm 2, ...$ The *angular momentum* of a particle is defined as $\mathcal{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the 'radius' of the circle, and $\mathbf{p}$ is the linear momentum.

(a) Show that the angular momentum of an electron in state $\psi_n(x)$ is $\mathcal{L}_n = n\hbar$, where $\hbar = \frac{h}{2\pi}$ is the 'reduced' Planck's constant. This implies that the angular momentum is *quantized* to values $0, \pm\hbar, \pm 2\hbar, ...$ Compare the quantized angular momentum $\mathcal{L}_1$ for $n = +1$ with the classical angular momentum $\mathcal{L}_{cl}$ of a mass $m = 1$ kg being spun by a string of length $R = 1$ m with tangential velocity $v = 1$ m/s to appreciate how 'nano' is the quantum of angular momentum.

(b) By balancing the classical centrifugal force and the electromagnetic Lorentz force, show that for an electron to be in the quantum state $\psi_n(x)$ on the ring, we need a *magnetic* field B_n such that the magnetic flux is $\Phi_n = B_n \cdot A = n \times \frac{h}{2e}$. Here A is the area of the ring, e is the electron charge and $h = 2\pi\hbar$. $\Phi_0 = \frac{h}{2e}$ is known as the *quantum of magnetic flux*, and has been measured experimentally in nanostructured rings.

(c) Consider the quantum state obtained by the superposition $\psi(x) = a[\psi_{n=1}(x) + \psi_{n=-1}(x)]$ from the eigenstates of the electron on the ring. Normalize the state to find the constant a. You may need the result $\int_0^L \cos^2(\frac{2\pi}{L}x)dx = \frac{L}{2}$. Does this superposition state have a definite momentum?

(d) We derived that the quantum expression for current flux is $\mathbf{j} = \frac{1}{2m}(\psi^\star \hat{\mathbf{p}} \psi - \psi \hat{\mathbf{p}} \psi^\star)$, where $\hat{\mathbf{p}} = -i\hbar\nabla$ is the momentum operator, which takes the form $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ for the particle on the ring. Show that even though the states $\psi_{n=1}(x)$ and $\psi_{n=-1}(x)$ carry *net* currents, their superposition state of part (c) does not. Explain.

2 Density of States, Fermi-Dirac distribution [30 points]

The electrons in the conduction band of graphene are free to move in 2-dimensions, forming a 2-dimensional electron gas (2DEG). The energy-momentum dispersion relationship for the 2DEG electrons in graphene is $E(k_x, k_y) = \hbar v_F \sqrt{k_x^2 + k_y^2}$, where v_F is a parameter with dimensions of velocity.

(a) Make a sketch of the energy as a function of the (k_x, k_y) points in the 2D k-space plane, and show that the dispersion results in a conical shape.

(b) Show that the density of states for these electrons is $g(E) = \frac{g_s g_v}{2\pi(\hbar v_F)^2}|E|$, where $g_s = 2$ is the spin degeneracy of each (k_x, k_y) state, and g_v is the number of cones in the energy dispersion. For graphene, $g_v = 2$.

(c) Show that at thermal equilibrium, when the Fermi level is at $E_f = 0$, the number of conduction electrons per unit area in 2D graphene is $n_i = \frac{\pi}{6}(\frac{kT}{\hbar v_F})^2$. You may need the result $\int_0^\infty dE \frac{E}{1+\exp\left(\frac{E}{\beta}\right)} = \frac{\pi^2\beta^2}{12}$.

3 Wigner-Seitz Cells and Reciprocal Lattice [30 points]

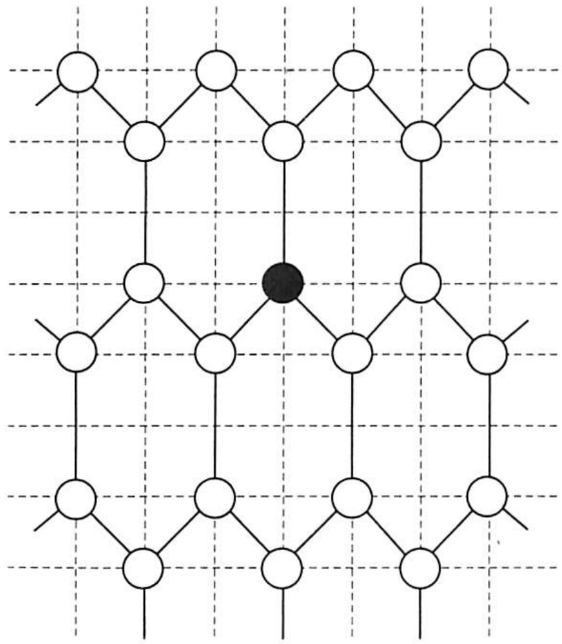

Figure 1: Atomic arrangement of a fictitious 2-dimensional crystal

Figure 1 shows the arrangement of atoms of a fictitious 2-dimensional crystal. All circles (filled and empty) represent the same atom.

(a) Using the filled 'atom' shown in black as the origin of the lattice, indicate the primitive translation vectors of this lattice. (There are several possible choices; use the simplest one, and explain.) How many atoms are there per lattice point?

(b) Determine the reciprocal lattice vectors corresponding to this lattice, and sketch the reciprocal lattice as accurately as you can. Use the grid lines shown as an unit of measure.

(c) Construct and show the 'Wigner-Seitz cell', or equivalently, the Brillouin zone clearly in the reciprocal lattice plot.