Dynamics of Flight Vehicles

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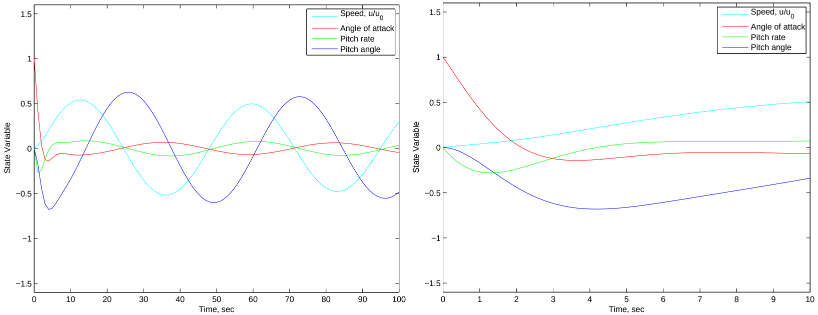
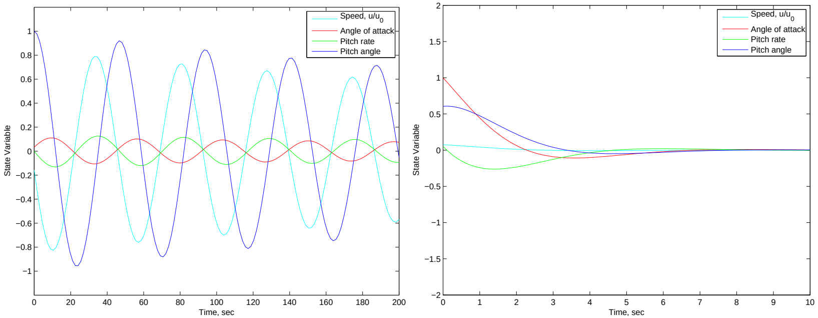


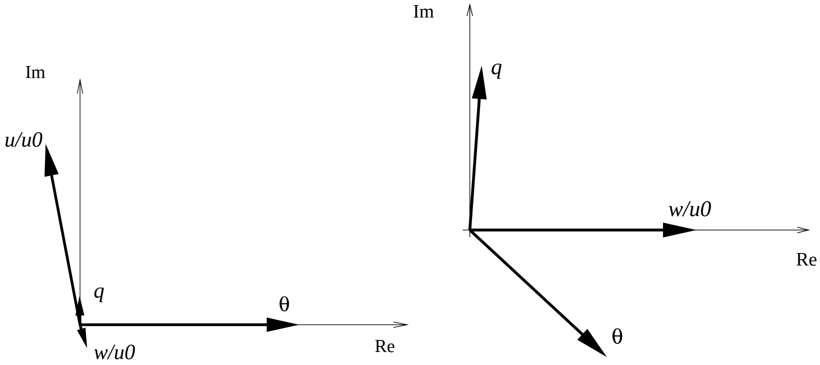
Figure 1: Response of Boeing 747 in powered approach to perturbation in angle of attack. Level flight at $\mathbf{M}_\infty = 0.25$ at standard sea level conditions. (Heffley's Condition 2.)



(a) Phugoid

(b) Short Period

Figure 2: Response of Boeing 747 in powered approach to perturbation in angle of attack. Level flight at $\mathbf{M}_\infty = 0.25$ at standard sea level conditions. (Heffley's Condition 2.)



(a) Phugoid (b) Short Period

Figure 3: Phasor plots for response of Boeing 747 in powered approach. Level flight at $\mathbf{M}_{\infty} = 0.25$ at standard sea level conditions. (Heffley's Condition 2.)

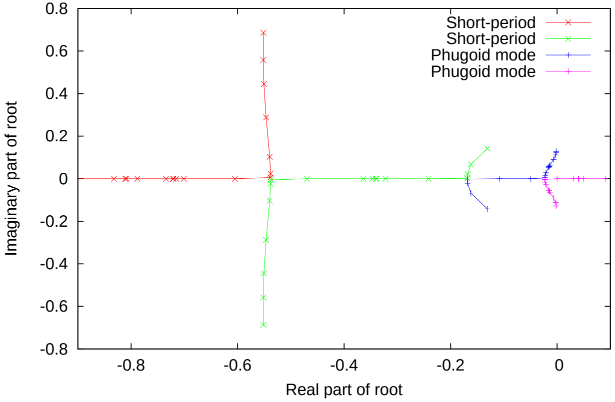


Figure 4: Locus of roots of characteristic equation for response of Boeing 747 in powered approach. Level flight at $\mathbf{M}_\infty = 0.25$ at standard sea level conditions. (Heffley's Condition 2.) As static margin is reduced from 0.22 to -.05, short-period roots join on real axis at 0.0158, phugoid roots join on real axis at 0.0021, one of the phugoid roots becomes neutrally stable at 0.0, and third oscillatory mode develops at -.0145.

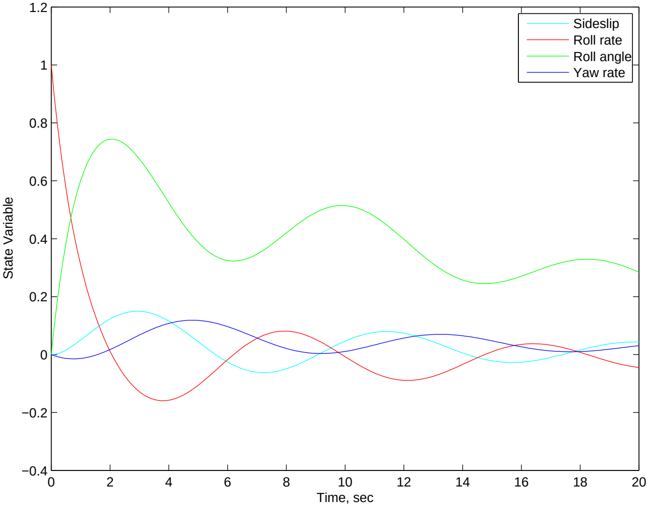
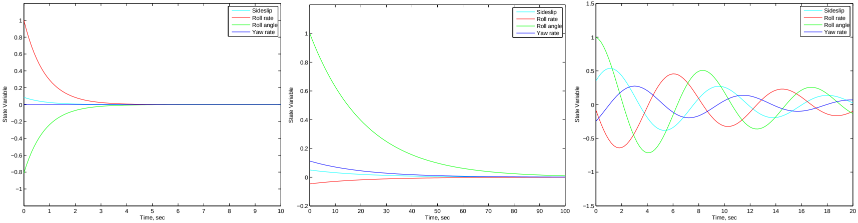
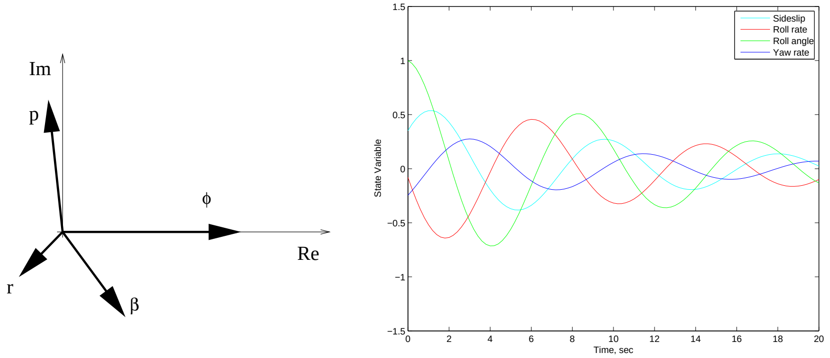


Figure 5: Response of Boeing 747 aircraft to unit perturbation in roll rate. Powered approach at $\mathbf{M}_\infty = 0.25$ at standard sea level conditions (Heffley's Condition 2).



(a) Rolling mode (b) Spiral mode (c) Dutch Roll mode

Figure 6: Response of Boeing 747 aircraft to unit perturbation in eigenvectors corresponding to the three lateral/directional modes. Powered approach at $\mathbf{M}_\infty = 0.25$ at standard sea level conditions (Heffley's Condition 2).



(a) Phasor plot (b) Response

Figure 7: Dutch Roll response of Boeing 747 aircraft in powered approach at $\mathbf{M}_\infty = 0.25$ at standard sea level conditions (Heffley's Condition 2). (a) Phasor plot, and (b) Response to initial perturbation exciting only the Dutch Roll mode.

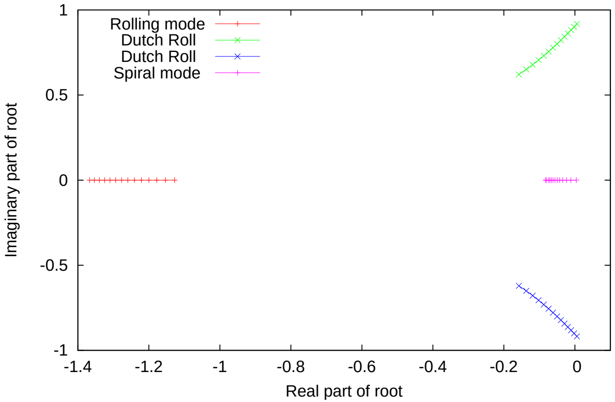


Figure 8: Locus of roots of plant matrix for Boeing 747 aircraft in powered approach at $\mathbf{M}_\infty = 0.25$ under standard sea level conditions (Heffley's Condition 2). Dihedral effect is varied from -.041 to -.561 in steps of -.04, while all other stability derivatives are held fixed at their nominal values. Rolling and spiral modes become increasingly stable as dihedral effect is increased; spiral mode becomes stable at $\mathbf{C}_{l\beta} \approx -.051$. Dutch Roll mode becomes less stable as dihedral effect is increased and becomes unstable at $\mathbf{C}_{l\beta} \approx -.532$.

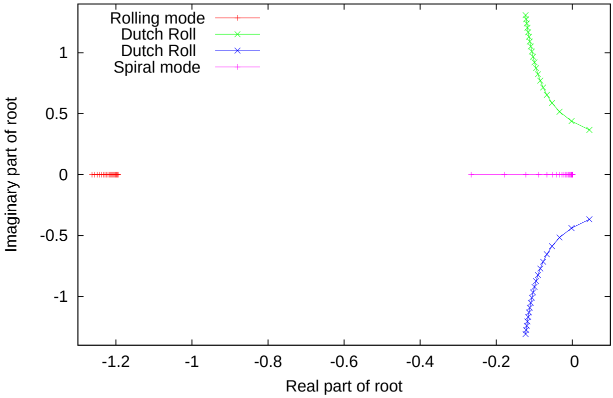


Figure 9: Locus of roots of plant matrix for Boeing 747 aircraft in powered approach at $\mathbf{M}_\infty = 0.25$ under standard sea level conditions (Heffley's Condition 2). Weathercock stability is varied Rom -.07 to 0.69 in steps of 0.04, while all other stability derivatives are held fixed at their nominal values. Rolling and spiral modes become less stable as weathercock stability is increased; spiral mode becomes unstable at $\mathbf{C}_{n\,\beta} \approx 0.657$. Dutch Roll mode becomes increasingly stable as weathercock stability is increased, but is unstable for less than $\mathbf{C}_{n\,\beta} \approx -.032$.

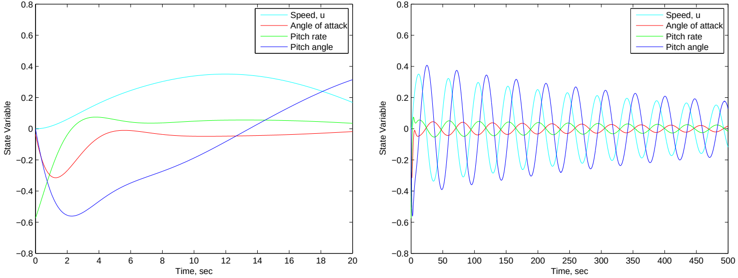


Figure 10: Response of Boeing 747 for powered approach at $\mathbf{M}_\infty = 0.25$ and standard sea level conditions to impulsive elevator input. Left plot is scaled to illustrate short-period response, and right plot is scaled to illustrate phugoid.

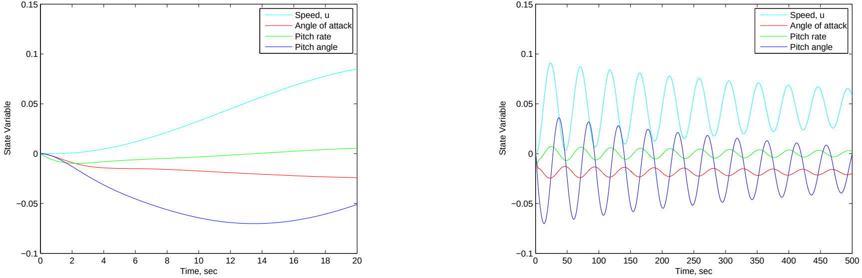


Figure 11: Response of Boeing 747 in powered approach at $\mathbf{M}_\infty = 0.25$ and standard sea level conditions to one-degree step elevator input. Left plot is scaled to illustrate short-period response, and right plot is scaled to illustrate phugoid.

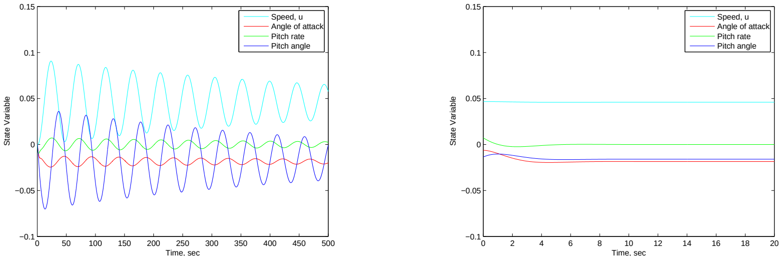


Figure 12: Response of Boeing 747 in powered approach at $\mathbf{M}_\infty = 0.25$ and standard sea level conditions to one-degree step elevator input. Left plot illustrates phugoid resulting from step input; right plot adds perturbation to initial condition to cancel transient phugoid component of step input.

System response to step input is

$$\mathbf{x} = \left(e^{\mathbf{A}t} - \mathbf{I}\right) \mathbf{A}^{-1}\mathbf{B}\eta_0 = e^{\mathbf{A}t}\left(\mathbf{A}^{-1}\mathbf{B}\eta_0\right) - \mathbf{A}^{-1}\mathbf{B}\eta_0 \qquad (1)$$

So, system ultimately settles into the new equilibrium state given by

$$\lim_{t\to\infty} \mathbf{x}(t) = -\mathbf{A}^{-1}\mathbf{B}\eta_0 = [0.0459 \quad -.0186 \quad 0.0 \quad -.0160]^T \qquad (2)$$

for the one-degree value of η_0. The new equilibrium state corresponds to an increase in flight speed at a reduced angle of attack, and the aircraft has begun to descend.

Longitudinal Control

Response to Step Elevator Input

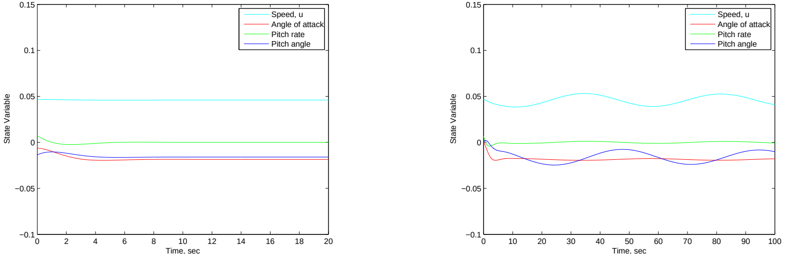


Figure 13: Response of Boeing 747 in powered approach at $\mathbf{M}_\infty = 0.25$ and standard sea level conditions to one-degree step elevator input. Left plot illustrates phugoid resulting from step input with perturbation added to the initial condition to cancel transient phugoid component of step input. Right plot illustrates degree to which the phugoid can be canceled by impulsive control input.

For one-degree elevator step input, the system ultimately settles into the new equilibrium state given by

$$\lim_{t \longrightarrow \infty} \mathbf{x}(t) = -\mathbf{A}^{-1}\mathbf{B}\eta_0 = [0.0459 \quad -.0186 \quad 0.0 \quad -.0160]^T \tag{3}$$

The phugoid excitation can be reduced by setting the initial condition to the phugoid component of the final state

$$\mathbf{x}_{\text{Ph}} = -\mathbf{Q}^{-1}\mathbf{A}^{-1}\mathbf{B}\eta_0 \tag{4}$$

or by including an impulsive control input designed to produce that initial perturbation (to the extent possible by the control available) from the least-squares approximation to the solution of

$$\mathbf{B}\eta_{\text{imp}} = \mathbf{x}_{\text{Ph}} \tag{5}$$

This result is completely consistent with that of our study of *static* longitudinal control, where the control sensitivity was found to be

$$\left.\frac{\mathrm{d}\delta_e}{\mathrm{d}\mathbf{C}_L}\right)_{\text{trim}} = \frac{\mathbf{C}_{m\alpha}}{\Delta} \tag{6}$$

where

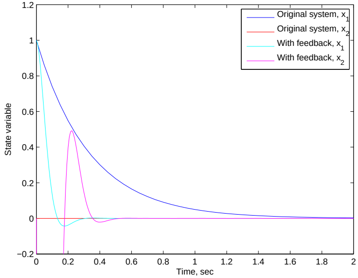
$$\Delta = -\mathbf{C}_{L\alpha}\mathbf{C}_{m\delta_e} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\delta_e} \tag{7}$$

Thus, from the static analysis we estimate for a step input of one degree in elevator

$$\Delta\mathbf{C}_L = \frac{\delta_e}{\mathbf{C}_{m\alpha}/\Delta} = \frac{\pi/180}{(-1.26)/(7.212)} = -.100 \tag{8}$$

The asymptotic steady state of the dynamic analysis gives exactly the same result

$$\Delta\mathbf{C}_L = \mathbf{C}_{L\alpha}\alpha + \mathbf{C}_{L\delta_e}\delta_e = 5.70(-.0186) + 0.338(\pi/180) = -.100 \tag{9}$$



(a) $\mathbf{x}(0) = [1.0 \quad 0.0]$ (b) $\mathbf{x}(0) = [1.0 \quad 0.1]$

Figure 14: Response of linear, second-order system, showing effect of state variable feedback. Original system has $\lambda_1 = 0.0$, $\lambda_2 = -3.0$. Modified system has $\omega_n = 25\ \text{sec}^{-1}$ and $\zeta = 0.707$. (a) $\mathbf{x}(0) = [1.0 \quad 0.0]^T$; (b) $\mathbf{x}(0) = [1.0 \quad 0.1]^T$.

Eigenvalues of original plant matrix are $\lambda_1 = 0$ and $\lambda_2 = 3$; corresponding eigenvectors are

$$\mathbf{u}_1 = [1 \quad 3/8]^T \qquad \text{and} \qquad \mathbf{u}_2 = [1 \quad 0]^T$$

so first initial condition has no $\mathbf{u}_2$ component. Final state of response of original system to second initial condition is parallel to first eigenvector.

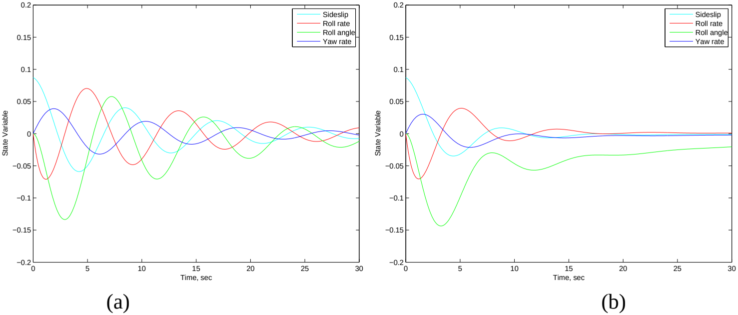


Figure 15: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M}_{\infty} = 0.25$; response to 5 degree (0.08727 radian) perturbation in sideslip. (a) Original open-loop response; (b) Closed loop response with Dutch Roll damping ratio changed to $\zeta = 0.30$ using rudder state-variable feedback.

The state and control vectors for lateral-directional motions are

$$\mathbf{x} = [v \quad p \quad \phi \quad r]^T \qquad \text{and} \qquad \eta = [\delta_r \quad \delta_a]^T \tag{10}$$

and the plant and control matrices are given by

$$\mathbf{A} = \begin{pmatrix} Y_v & Y_p & g\cos\Theta_0 & Y_r - u_0 \\ \frac{L_v + i_x N_v}{1 - i_x i_z} & \frac{L_p + i_x N_p}{1 - i_x i_z} & 0 & \frac{L_r + i_x N_r}{1 - i_x i_z} \\ 0 & 1 & 0 & 0 \\ \frac{N_v + i_z L_v}{1 - i_x i_z} & \frac{N_p + i_z L_p}{1 - i_x i_z} & 0 & \frac{N_r + i_z L_r}{1 - i_x i_z} \end{pmatrix} \tag{11}$$

and

$$\mathbf{B} = \begin{pmatrix} Y_{\delta_r} & 0 \\ \frac{L_{\delta_r} + i_x N_{\delta_r}}{1 - i_x i_z} & \frac{L_{\delta_a} + i_x N_{\delta_a}}{1 - i_x i_z} \\ 0 & 0 \\ \frac{N_{\delta_r} + i_z L_{\delta_r}}{1 - i_x i_z} & \frac{N_{\delta_a} + i_z L_{\delta_a}}{1 - i_x i_z} \end{pmatrix} \tag{12}$$

respectively.
The original plant matrix and the *changes* to the plant matrix for improved Dutch-Roll damping due to *rudder-only* state variable feedback control are

$$\mathbf{A} = \begin{pmatrix} -.0999 & 0.0000 & 0.1153 & -1.000 \\ -1.604 & -1.0932 & 0.0000 & 0.2850 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.4089 & -.0395 & 0.0000 & -.2454 \end{pmatrix} \quad \text{and} \quad \Delta\mathbf{A} = \begin{pmatrix} -.0025 & -.0017 & -.0023 & 0.0206 \\ -.0120 & -.0082 & -.0109 & 0.0984 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0337 & 0.0230 & 0.0305 & -.2766 \end{pmatrix} \tag{13}$$

respectively.

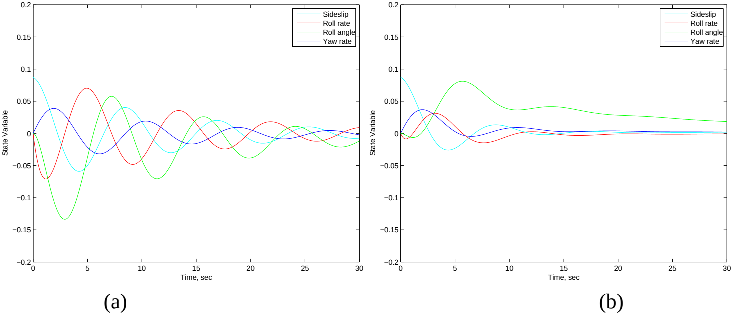


Figure 16: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M}_\infty = 0.25$; response to 5 degree (0.08727 radian) perturbation in sideslip. (a) Original open-loop response; (b) Closed loop response with Dutch Roll damping ratio changed to $\zeta = 0.30$ using aileron state-variable feedback.

The state and control vectors for lateral-directional motions are

$$\mathbf{x} = [v \quad p \quad \phi \quad r]^T \qquad \text{and} \qquad \eta = [\delta_r \quad \delta_a]^T \tag{14}$$

and the plant and control matrices are given by

$$\mathbf{A} = \begin{pmatrix} Y_v & Y_p & g\cos\Theta_0 & Y_r - u_0 \\ \frac{L_v + i_x N_v}{1 - i_x i_z} & \frac{L_p + i_x N_p}{1 - i_x i_z} & 0 & \frac{L_r + i_x N_r}{1 - i_x i_z} \\ 0 & 1 & 0 & 0 \\ \frac{N_v + i_z L_v}{1 - i_x i_z} & \frac{N_p + i_z L_p}{1 - i_x i_z} & 0 & \frac{N_r + i_z L_r}{1 - i_x i_z} \end{pmatrix} \tag{15}$$

and

$$\mathbf{B} = \begin{pmatrix} Y_{\delta_r} & 0 \\ \frac{L_{\delta_r} + i_x N_{\delta_r}}{1 - i_x i_z} & \frac{L_{\delta_a} + i_x N_{\delta_a}}{1 - i_x i_z} \\ 0 & 0 \\ \frac{N_{\delta_r} + i_z L_{\delta_r}}{1 - i_x i_z} & \frac{N_{\delta_a} + i_z L_{\delta_a}}{1 - i_x i_z} \end{pmatrix} \tag{16}$$

respectively.

The original plant matrix and the *changes* to the plant matrix for improved Dutch-Roll damping due to *aileron-only* state variable feedback control are

$$\mathbf{A} = \begin{pmatrix} -.0999 & 0.0000 & 0.1153 & -1.000 \\ -1.604 & -1.0932 & 0.0000 & 0.2850 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.4089 & -.0395 & 0.0000 & -.2454 \end{pmatrix} \quad \text{and} \quad \Delta\mathbf{A} = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.1387 & -.2802 & -.2169 & 1.3023 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -.0062 & 0.0015 & 0.0012 & -.0071 \end{pmatrix} \tag{17}$$

respectively.

The Bass-Gura procedure often problematic for a variety of reasons:

- Best placement for the eigenvalues of the augmented matrix is not always obvious;
- Particular eigenvalue placement may require more control input than is available;
- For multiple input-output systems, we need strategies for how best to allocate the gains among the $n \times p$ elements, since we have only n eigenvalues to place;
- The process may not be controllable; i.e., if the rank of the controllability matrix $\mathbf{V}$ is less than n, the method fails.

All these points argue for a control design strategy that, in some sense, optimizes the gain matrix for stabilizing a given system. This is the goal of what has come to be called *optimal control*.

The *optimal control* of the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta(t) \tag{18}$$

is defined as the control vector $\eta(t)$ that drives the state from a specified initial state $\mathbf{x}(t)$ to a desired final state $\mathbf{x}_d(t_f)$ such that a specified performance index

$$J = \int_t^{t_f} g(\mathbf{x}(\tau), \eta(\tau), \tau)\, \mathrm{d}\tau \tag{19}$$

is minimized. For *quadratic* optimal control, the performance index is specified in the form

$$g = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \eta^T\mathbf{R}\eta \tag{20}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are symmetric, positive-definite matrices, and the performance index becomes

$$J = \int_t^{t_f} \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \eta^T\mathbf{R}\eta\right)\, \mathrm{d}\tau \tag{21}$$

If the control law is assumed to be linear, i.e., of the form

$$\eta = -\mathbf{K}\mathbf{x} + \eta' \tag{22}$$

then the determination of the gain matrix $\mathbf{K}$ that minimizes J is called the linear quadratic regulator (LQR) problem.

For this control law the closed-loop response of the system to a perturbation is given by

$$\dot{\mathbf{x}} = [\mathbf{A} - \mathbf{B}\mathbf{K}]\,\mathbf{x} = \mathbf{A}^*\mathbf{x} \tag{23}$$

where $\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{K}$ is the *augmented* plant matrix.
We usually are interested in cases for which the matrices **A**, **B**, and **K** are independent of time, but the development here is easier if we allow the augmented matrix $\mathbf{A}^*$ to vary with time. In this case, we need to express the solution to Eq. (23) in terms of the general state transition matrix Φ^* as

$$\mathbf{x}(\tau) = \Phi^*(\tau, t)\mathbf{x}(t) \tag{24}$$

Equation (24) simply implies that the state of the system at any time τ depends linearly on the state at any other time t. When the control law of Eq. (22) is substituted and Eq. (24) is used to express the evolution of the state variable, the performance index of Eq. (21) becomes

$$\begin{aligned} J &= \int_t^{t_f} \mathbf{x}^T(\tau)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\mathbf{x}(\tau)\,\mathrm{d}\tau \\ &= \int_t^{t_f} \mathbf{x}^T(t)\Phi^{*T}(\tau, t)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\Phi^*(\tau, t)\mathbf{x}(t)\,\mathrm{d}\tau \\ &= \mathbf{x}^T(t)\left(\int_t^{t_f} \Phi^{*T}(\tau, t)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\Phi^*(\tau, t)\,\mathrm{d}\tau\right)\mathbf{x}(t) \end{aligned} \tag{25}$$

or

$$J = \mathbf{x}^T(t)\mathbf{S}\mathbf{x}(t) \tag{26}$$

where

$$\mathbf{S}(t, t_f) = \int_t^{t_f} \Phi^{*T}(\tau, t)\left[\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K}\right]\Phi^*(\tau, t)\,\mathrm{d}\tau \tag{27}$$

Note that, by its construction, the matrix **S** is symmetric, since the weighting matrices **Q** and **R** are both symmetric.

The simple appearance of Eq. (26) belies the complexity of determining **S** from Eq. (27) because it is almost impossible to develop a useful expression for the state transition matrix in general. Instead, in order to find the gain matrix **K** that minimizes J, it is convenient to find a differential equation that the matrix **S** satisfies. To this end, we note that since

$$J = \int_t^{t_f} \mathbf{x}^T(\tau)\mathbf{L}\mathbf{x}(\tau)\,\mathrm{d}t \tag{28}$$

where

$$L = \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K} \tag{29}$$

we can write

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\left.\mathbf{x}^T(\tau)\mathbf{L}\mathbf{x}(\tau)\right|_{\tau=t} = -\mathbf{x}^T(t)\mathbf{L}\mathbf{x}(t) \tag{30}$$

But, from differentiating Eq. (26), we have

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \dot{\mathbf{x}}^T(t)\mathbf{S}(t,t_f)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{\mathbf{S}}(t,t_f)\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{S}(t,t_f)\dot{\mathbf{x}}(t) \tag{31}$$

and, substituting the closed-loop differential equation, Eq. (23), for $\dot{\mathbf{x}}$ gives

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \mathbf{x}^T(t)\left[\mathbf{A}^{*T}\mathbf{S}(t,t_f) + \dot{\mathbf{S}}(t,t_f) + \mathbf{S}(t,t_f)\mathbf{A}^*(t)\right]\mathbf{x}(t) \tag{32}$$

Thus, we have two expressions for the derivative $\mathrm{d}J/\mathrm{d}t$: Eqs. (30) and (32). Both are quadratic forms in the initial state $\mathbf{x}(t)$, which must be *arbitrary*. The only way that two quadratic forms in **x** can be equal for any choice of **x** is if the underlying matrices are equal; thus, we must have

$$-\dot{\mathbf{S}} = \mathbf{S}\mathbf{A}^* + \mathbf{A}^{*T}\mathbf{S} + \mathbf{L} \tag{33}$$

Equation (33)

$$-\dot{\mathbf{S}} = \mathbf{S}\mathbf{A}^* + \mathbf{A}^{*T}\mathbf{S} + \mathbf{L} \tag{34}$$

is a first-order differential equation for the matrix **S**, so it requires a single initial condition to completely specify its solution. We can use Eq. (27), evaluated at $t = t_f$ to give the required condition

$$\mathbf{S}(t_f, t_f) = 0 \tag{35}$$

Once a gain matrix **K** has been chosen to close the loop, the corresponding performance of the system is given by Eq. (26), where $\mathbf{S}(t, t_f)$ is the solution of Eq. (33), which can be written in terms of the original plant and gain matrices as

$$-\dot{\mathbf{S}} = \mathbf{S}\left(\mathbf{A} - \mathbf{B}\mathbf{K}\right) + \left(\mathbf{A}^T - \mathbf{K}^T\mathbf{B}^T\right)\mathbf{S} + \mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K} \tag{36}$$

Our task, then, is to find the gain matrix **K** that makes the solution to Eq. (36) as small as possible – in the sense that the quadratic forms (Eq. (26)) associated with the matrix **S** are minimized. That is, we want to find the matrix $\hat{\mathbf{S}}$ for which

$$\hat{J} = \mathbf{x}^T\hat{\mathbf{S}}\mathbf{x} < \mathbf{x}^T\mathbf{S}\mathbf{x} \tag{37}$$

for any arbitrary initial state $\mathbf{x}(t)$ and every matrix $\mathbf{S} \neq \hat{\mathbf{S}}$.
We will proceed by assuming that such an optimum exists, and use the calculus of variations to find it. The minimizing matrix $\hat{\mathbf{S}}$ must, of course, satisfy Eq. (36)

$$-\dot{\hat{\mathbf{S}}} = \hat{\mathbf{S}}\left(\mathbf{A} - \mathbf{B}\hat{\mathbf{K}}\right) + \left(\mathbf{A}^T - \hat{\mathbf{K}}^T\mathbf{B}^T\right)\hat{\mathbf{S}} + \mathbf{Q} + \hat{\mathbf{K}}^T\mathbf{R}\hat{\mathbf{K}} \tag{38}$$

and any *non*-optimum gain matrix, and its corresponding matrix **S**, can be expressed as

$$\mathbf{S} = \hat{\mathbf{S}} + \mathbf{N} \qquad \text{and} \qquad \mathbf{K} = \hat{\mathbf{K}} + \mathbf{Z} \tag{39}$$

Substituting this form into Eq. (36) and subtracting Eq. (38) gives

$$-\dot{\mathbf{N}} = \mathbf{N}\mathbf{A}^* + \mathbf{A}^{*T}\mathbf{N} + \left(\hat{\mathbf{K}}^T\mathbf{R} - \hat{\mathbf{S}}\mathbf{B}\right)\mathbf{Z} + \mathbf{Z}^T\left(\mathbf{R}\hat{\mathbf{K}} - \mathbf{B}^T\hat{\mathbf{S}}\right) + \mathbf{Z}^T\mathbf{R}\mathbf{Z} \tag{40}$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{K} = \mathbf{A} - \mathbf{B}\left(\hat{\mathbf{K}} + \mathbf{Z}\right) \tag{41}$$

Note that Eq. (40) has exactly the same form as Eq. (33) with

$$\mathbf{L} = \left(\hat{\mathbf{K}}^T\mathbf{R} - \hat{\mathbf{S}}\mathbf{B}\right)\mathbf{Z} + \mathbf{Z}^T\left(\mathbf{R}\hat{\mathbf{K}} - \mathbf{B}^T\hat{\mathbf{S}}\right) + \mathbf{Z}^T\mathbf{R}\mathbf{Z} \tag{42}$$

so its solution must be of the form of Eq. (27)

$$\mathbf{N}(t, t_f) = \int_t^{t_f} \Phi^{*T}(\tau, t)\mathbf{L}\Phi^*(\tau, t)\,\mathrm{d}\tau \tag{43}$$

Now, if $\hat{J}$ is a minimum, then we must have

$$\mathbf{x}^T\hat{\mathbf{S}}\mathbf{x} \leq \mathbf{x}^T\left(\hat{\mathbf{S}} + \mathbf{N}\right)\mathbf{x} = \mathbf{x}^T\hat{\mathbf{S}}\mathbf{x} + \mathbf{x}^T\mathbf{N}\mathbf{x} \tag{44}$$

and this equation requires that the quadratic form $\mathbf{x}^T\mathbf{N}\mathbf{x}$ be positive definite (or, at least, positive semi-definite). But, if $\mathbf{Z}$ is sufficiently small, the linear terms in $\mathbf{Z}$ (and $\mathbf{Z}^T$) in Eq. (42) will dominate the quadratic terms in $\mathbf{Z}^T\mathbf{R}\mathbf{Z}$, and we could easily find values of $\mathbf{Z}$ that would make $\mathbf{L}$, and hence $\mathbf{N}$, negative definite. Thus, *the linear terms in Eq. (42) must be absent altogether*. That is, for the gain matrix $\hat{\mathbf{K}}$ to be optimum, we must have

$$\hat{\mathbf{K}}^T\mathbf{R} - \hat{\mathbf{S}}\mathbf{B} = 0 = \mathbf{R}\hat{\mathbf{K}} - \mathbf{B}^T\hat{\mathbf{S}} \tag{45}$$

or, assuming that the weighting matrix $\mathbf{R}$ is not singular,

$$\hat{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}^T\hat{\mathbf{S}} \tag{46}$$

Equation (46) gives the optimum gain matrix $\hat{\mathbf{K}}$, once the matrix $\hat{\mathbf{S}}$ has been determined. When this equation is substituted back into Eq. (38) we have

$$-\dot{\hat{\mathbf{S}}} = \hat{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\hat{\mathbf{S}} - \hat{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\hat{\mathbf{S}} + \mathbf{Q} \tag{47}$$

This equation, one of the most famous in modern control theory, is called the *matrix Riccati equation*, consistent with the mathematical nomenclature that identifies an equation with a quadratic non-linearity as a Riccati equation. The solution to this equation gives the matrix $\hat{\mathbf{S}}$ which, when substituted into Eq. (46), gives the optimum gain matrix $\hat{\mathbf{K}}$.

Due to the quadratic nonlinearity of the Riccati equation, it usually is necessary to solve it numerically. Since the matrix $\hat{\mathbf{S}}$ is symmetric, Eq. (47) represents $n(n+1)/2$ coupled, first-order equations. Since the "initial" condition is

$$\hat{\mathbf{S}}(t_f, t_f) = 0 \tag{48}$$

the equation must be integrated *backward* in time, since we are interested in $\hat{\mathbf{S}}(t, t_f)$ for $t < t_f$. When the control interval $[t, t_f]$ is finite, the gain matrix $\mathbf{K}$ is generally time-dependent, even if the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, and $\mathbf{R}$ all are constant. But, suppose the control interval is *infinite*, so we want to find the gain matrix $\hat{\mathbf{K}}$ that minimizes the performance index

$$J_\infty = \int_t^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \eta^T \mathbf{R} \eta \right) \, d\tau \tag{49}$$

In this case, integration of Eq. (47) backward in time will either grow without limit or converge to a *constant* matrix $\bar{\mathbf{S}}$. If it converges to a limit, the derivative $\dot{\hat{\mathbf{S}}}$ must tend to zero, and $\bar{\mathbf{S}}$ must satisfy the *algebraic* equation

$$0 = \bar{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} + \mathbf{Q} \tag{50}$$

and the optimum gain in the steady state is given by

$$\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} \tag{51}$$

The single *quadratic* matrix Eq. (50) represents $n(n+1)/2$ coupled scalar, quadratic equations, so we expect $n(n+1)$ different (symmetric) solutions. The nature of these solutions is connected with issues of controllability and observability; for our purposes here, it is enough to know that

- If the system is asymptotically stable; or
- If the system defined by $(\mathbf{A}, \ \mathbf{B})$ is *controllable*, and the system defined by $(\mathbf{A}, \ \mathbf{C})$, where the weighting matrix $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$, is observable,

then the algebraic Riccati equation has an unique positive definite solution $\bar{\mathbf{S}}$ that minimizes J_∞ when the control law

$$\eta = -\bar{\mathbf{K}}\mathbf{x} = -\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}}\mathbf{x} \tag{52}$$

is used. (Note that there are still $n(n+1)$ symmetric solutions; the assertion here is that, of these multiple solutions, one, and only one, is *positive definite*.)

We consider using optimal control to stabilize an inverted pendulum. The equation of motion for an inverted pendulum near its (unstable) equilibrium point, as illustrated in Fig. 17 is

$$mL^2\ddot{\theta} = mgL\sin\theta + T = mgL\theta + T \tag{53}$$

where m is the mass of the pendulum, L is the pendulum length, g is the acceleration of gravity, and T is the externally-applied (control) torque; the second form of the right-hand side assumes the angle θ is small.

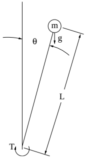


Figure 17: Inverted pendulum affected by gravity g and control torque T.

If we introduce the angular velocity $\omega = \dot{\theta}$ as a second state variable, Eq. (53) can be written in the standard state variable form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta \\ \omega\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \gamma & 0\end{pmatrix}\begin{pmatrix}\theta \\ \omega\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix}\tau \tag{54}$$

where $\gamma = g/L$ and $\tau = T/(mL^2)$ are reduced gravity and input torque variables.
Now, we seek the control law that minimizes the performance index

$$J_\infty = \int_t^\infty \left(\theta^2 + \frac{\tau^2}{c^2}\right)\,\mathrm{d}t' \tag{55}$$

where c is a parameter that determines the relative weighting of control input and angular deviation in the penalty function. It is clear that this performance index corresponds to

$$\mathbf{Q} = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix} \quad \text{and} \quad \mathbf{R} = \frac{1}{c^2} \tag{56}$$

If we define the elements of the matrix $\bar{\mathbf{S}}$ to be

$$\bar{\mathbf{S}} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \tag{57}$$

then the optimum gain matrix is

$$\bar{\mathbf{K}} = \mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} = c^2 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} = \begin{bmatrix} c^2 s_2 & c^2 s_3 \end{bmatrix} \tag{58}$$

which is seen to be independent of the element s_1. The terms needed for the algebraic Riccati equation

$$0 = \bar{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} + \mathbf{Q} \tag{59}$$

are

$$\bar{\mathbf{S}}\mathbf{A} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} s_2\gamma & s_1 \\ s_3\gamma & s_2 \end{pmatrix} \tag{60}$$

$$\mathbf{A}^T\bar{\mathbf{S}} = \begin{pmatrix} 0 & \gamma \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} = \begin{pmatrix} s_2\gamma & s_3\gamma \\ s_1 & s_2 \end{pmatrix} = (\bar{\mathbf{S}}\mathbf{A})^T \tag{61}$$

and

$$\bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} c^2 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} = c^2 \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} \tag{62}$$

Thus, the Riccati equation is

$$0 = \begin{pmatrix} s_2\gamma & s_1 \\ s_3\gamma & s_2 \end{pmatrix} + \begin{pmatrix} s_2\gamma & s_3\gamma \\ s_1 & s_2 \end{pmatrix} - c^2 \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{63}$$

which is equivalent to the three scalar equations

$$\begin{aligned} 0 &= 2s_2\gamma - c^2 s_2^2 + 1 \\ 0 &= s_1 + s_3\gamma - c^2 s_2 s_3 \\ 0 &= 2s_2 - c^2 s_3^2 \end{aligned} \tag{64}$$

These can be solved in closed form. The first of Eqs. (64) gives

$$s_2 = \frac{\gamma \pm \sqrt{\gamma^2 + c^2}}{c^2} \tag{65}$$

and the third of Eqs. (64) gives

$$s_3 = \pm \frac{1}{c}\sqrt{2s_2} \tag{66}$$

Since the elements of $\bar{\mathbf{S}}$ must be real, s_2 must be positive (or s_3 would be complex). Thus, we must choose the positive root in Eq. (65). Further, the second of Eqs. (64) gives

$$s_1 = c^2 s_2 s_3 - \gamma s_3 = s_3\sqrt{\gamma^2 + c^2} \tag{67}$$

Thus, elements s_1 and s_3 have the same sign which, for $\bar{\mathbf{S}}$ to be positive definite, must be positive. Thus,

$$\begin{aligned} s_2 &= \frac{\gamma + \sqrt{\gamma^2 + c^2}}{c^2} \\ s_3 &= \frac{1}{c}\sqrt{2s_2} = \frac{\sqrt{2}}{c^2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2} \end{aligned} \tag{68}$$

represents the unique solution for the corresponding elements for which $\bar{\mathbf{S}}$ is positive definite.

Thus, the (optimal) gain matrix is seen to be

$$\bar{\mathbf{K}} = \begin{bmatrix} c^2 s_2 & c^2 s_3 \end{bmatrix} = \left[\gamma + \sqrt{\gamma^2 + c^2} \quad \sqrt{2} \left[\gamma + \sqrt{\gamma^2 + c^2} \right]^{1/2} \right] \tag{69}$$

The augmented matrix is then given by

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\bar{\mathbf{K}} = \begin{pmatrix} 0 & 1 \\ -\sqrt{\gamma^2 + c^2} & -\sqrt{2} \left[\gamma + \sqrt{\gamma^2 + c^2} \right]^{1/2} \end{pmatrix} \tag{70}$$

and its characteristic equation is

$$\lambda^2 + \sqrt{2} \left[\gamma + \sqrt{\gamma^2 + c^2} \right]^{1/2} \lambda + \sqrt{\gamma^2 + c^2} = 0 \tag{71}$$

which has roots

$$\lambda = \frac{\sqrt{2}}{2} \left[-\sqrt{\gamma + \bar{\gamma}} \pm \imath \sqrt{\bar{\gamma} - \gamma} \right] \tag{72}$$

where we have introduced

$$\bar{\gamma} = \sqrt{\gamma^2 + c^2} \tag{73}$$

Note that as c/γ becomes large, $\bar{\gamma}$ becomes large relative to γ, so

$$\lim_{c/\gamma \to \infty} \lambda = -\sqrt{\frac{\bar{\gamma}}{2}}\,(1 \pm \imath) \qquad (74)$$

Thus, as c becomes large, the damping ratio of the system approaches a constant value of

$$\zeta = \frac{1}{\sqrt{2}}$$

while the undamped natural frequency increases as

$$\omega_n = \sqrt{\bar{\gamma}} \approx \sqrt{c}$$

Large values of c correspond to a performance index in which the weighting of the control term is small compared to that of the deviations in state variables – i.e., to a situation in which we are willing to spend additional energy in control to maintain very small perturbations of the state from its equilibrium position.

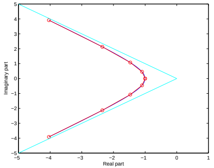


Figure 18: Locus of roots of characteristic equation of augmented plant matrix for inverted pendulum. Axes are scaled to give roots in units of γ. Open symbols represent roots at $c/\gamma = 0, 1, 10, 100, 1000$, with real root corresponding to $c/\gamma = 0$. Cyan lines represent asymptotes in the limit of large c/γ.

On the other hand, as c becomes small, the weighting of the control term in the performance index becomes large compared to that of the state variables. This is consistent with the fact that the gains in Eq. (69)

$$K_1 = \gamma + \sqrt{\gamma^2 + c^2} \qquad \text{and} \qquad K_2 = \sqrt{2}\left[\gamma + \sqrt{\gamma^2 + c^2}\right]^{1/2}$$

decrease monotonically with c. In the limit $c = 0$, however, the gains remain finite, with

$$\lim_{c \to 0} K_1 = 2\gamma \qquad \text{and} \qquad \lim_{c \to 0} K_2 = 2\sqrt{\gamma}$$

since *some* control is necessary to stabilize this, otherwise unstable, system.

We here apply linear, quadratic optimal control to improve the stability of the Boeing 747 aircraft in powered approach at $\mathbf{M} = 0.25$ at standard sea level conditions.
We apply linear, quadratic, optimal control to minimize the steady state performance index

$$J_\infty = \int_t^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \frac{1}{c^2} \eta^T \mathbf{R} \eta \right) \, \mathrm{d}\tau \tag{75}$$

where c is again a parameter that determines the relative weights given to control action and perturbations in the state variable in the penalty function. For lateral/directional motions at this flight condition, the plant matrix is the same as used earlier, while the control matrix is given by

$$\mathbf{B} = \begin{pmatrix} 0.0182 & 0.0868 & 0.0000 & -.2440 \\ 0.0000 & 0.3215 & 0.0000 & -.0017 \end{pmatrix}^T \tag{76}$$

where the control vector is

$$\eta = [\delta_r \quad \delta_a]^T \tag{77}$$

The weighting matrices in the performance index are taken to be simply

$$\mathbf{Q} = \mathbf{I} \quad \text{and} \quad \mathbf{R} = \mathbf{I} \tag{78}$$

where $\mathbf{Q}$ is a 4×4 matrix and $\mathbf{R}$ is a 2×2 matrix.
The MATLAB function

[S, L, G] = care(A,B,Q,R,T,E);

is used to solve the generalized matrix Riccati equation

$$\mathbf{E}^T \mathbf{S} \mathbf{A} + \mathbf{A}^T \mathbf{S} \mathbf{E} - \left(\mathbf{E}^T \mathbf{S} \mathbf{B} + \mathbf{T} \right) \mathbf{R}^{-1} \left(\mathbf{B}^T \mathbf{S} \mathbf{E} + \mathbf{T}^T \right) + \mathbf{Q} = 0 \tag{79}$$

which, with the additional input matrices are defined as

T = zeros(size(B)) and *E = eye(size(A))*

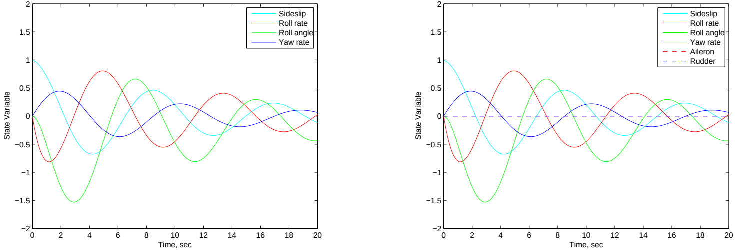
reduces to Eq. (50). In addition to the solution matrix *S*, the MATLAB function *care* also returns the gain matrix

$$\mathbf{G} = \mathbf{R}^{-1} \left(\mathbf{B}^T \mathbf{S} \mathbf{E} + \mathbf{T}^T \right) \tag{80}$$

and the vector

L = eig(A - BG, E)

containing the eigenvalues of the augmented matrix.

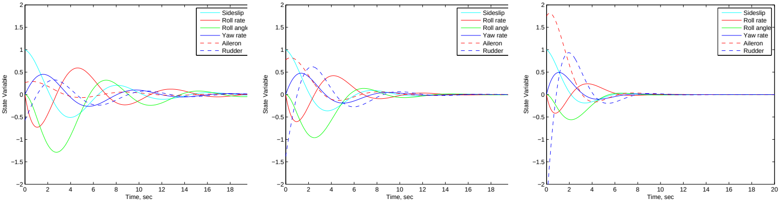


(a) No feedback (b) $c = 0.001$

Figure 19: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M}_\infty = 0.25$; response to unit perturbation in sideslip. (a) Original open-loop response; (b) Optimal closed loop response with performance parameter $c = 0.001$.

Lateral/Directional Feedback Control

Linear-Quadratic Control as a Stability Augmentation System



(a) $c = 0.50$ (b) $c = 1.0$ (c) $c = 2.0$

Figure 20: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M}_\infty = 0.25$; response to unit perturbation in sideslip illustrating effect of varying weighting parameter c. Optimal closed-loop responses with (a) $c = 0.50$; (b) $c = 1.0$; and (c) $c = 2.0$. Control deflections required to stabilize the motions are also shown.

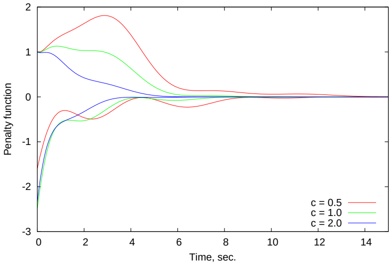


Figure 21: Penalty functions in performance index for optimal control solution; Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M}_\infty = 0.25$. Upper curves are $\mathbf{x}^T\mathbf{Q}\mathbf{x}$, and lower curves are $-\eta^T\mathbf{R}\eta$, as functions of time for response to unit perturbation in sideslip angle β, so the area between the curves is equal to the performance index J_∞.

The state and control vectors for lateral-directional motions are

$$\mathbf{x} = [v \quad p \quad \phi \quad r]^T \qquad \text{and} \qquad \eta = [\delta_r \quad \delta_a]^T \tag{81}$$

and the plant and control matrices are given by

$$\mathbf{A} = \begin{pmatrix} Y_v & Y_p & g\cos\Theta_0 & Y_r - u_0 \\ \frac{L_v + i_x N_v}{1 - i_x i_z} & \frac{L_p + i_x N_p}{1 - i_x i_z} & 0 & \frac{L_r + i_x N_r}{1 - i_x i_z} \\ 0 & 1 & 0 & 0 \\ \frac{N_v + i_z L_v}{1 - i_x i_z} & \frac{N_p + i_z L_p}{1 - i_x i_z} & 0 & \frac{N_r + i_z L_r}{1 - i_x i_z} \end{pmatrix} \tag{82}$$

and

$$\mathbf{B} = \begin{pmatrix} Y_{\delta_r} & 0 \\ \frac{L_{\delta_r} + i_x N_{\delta_r}}{1 - i_x i_z} & \frac{L_{\delta_a} + i_x N_{\delta_a}}{1 - i_x i_z} \\ 0 & 0 \\ \frac{N_{\delta_r} + i_z L_{\delta_r}}{1 - i_x i_z} & \frac{N_{\delta_a} + i_z L_{\delta_a}}{1 - i_x i_z} \end{pmatrix} \tag{83}$$

respectively.
The original plant matrix and the *changes* to the plant matrix for improved response using *LQC* with $c = 2.0$ are

$$\mathbf{A} = \begin{pmatrix} -.0999 & 0.0000 & 0.1153 & -1.000 \\ -1.604 & -1.0932 & 0.0000 & 0.2850 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.4089 & -.0395 & 0.0000 & -.2454 \end{pmatrix} \quad \text{and} \quad \Delta\mathbf{A} = \begin{pmatrix} -.0458 & -.0114 & 0.0099 & 0.0773 \\ 0.3433 & -.4532 & -.4662 & -.0253 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.6100 & -.1498 & -.1299 & -1.0334 \end{pmatrix} \tag{84}$$

respectively.

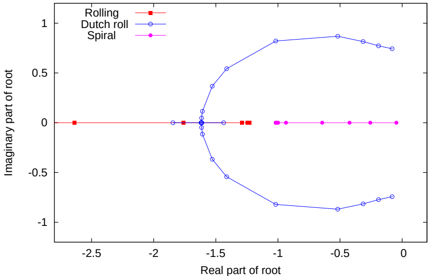


Figure 22: Boeing 747 aircraft in powered approach at standard sea level conditions and $\mathbf{M}_\infty = 0.25$; locus of roots of characteristic equation of augmented matrix as control weighting parameter c is increased. Symbols represent root locations for $c = 0.001, 0.5, 1.0, 2.0, 5.0, 8.0, 9.0, 9.7, 9.76, 9.7727, 9.7728, 10.0$; as c is increased, all roots move to the left (except for one of the Dutch Roll roots after that mode becomes critically damped between $9.7227 < c < 9.7228$).