February 15th
Expectation

Expectation

Suppose that a doctor get only two kinds of patients, those with insurance type `a' and one with insurance `b'. If a patient has insurance `a' the doctor gets 40 dollars per visit, if the patient is from insurance `b' he gets 55. Let INCOME be a random variable from the event space of insurance types to the space of dollars defined by $f(A) = 40$ and $f(B) = 55$. P(INCOME=a) is 1/3 and P(INCOME=b) is 2/3. How much money does the doctor get on average from every patient? The answer is

$1/3 * 40 + 2/3 * 55 = 50$

This value is known as the **expected value** or expectation of INCOME.

MEAN: the value of a random variable that you expect, on average

$$\mu_X = \sum_{\text{all possible values of} X} (\text{value of the R.V. } X) \times P(X)$$

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$$\mu = E(X)$$

Expectation Value

The expectation value of a function $f(x)$ in a variable x is denoted $\langle f(x) \rangle$ or $E\{f(x)\}$. For a single discrete variable, it is defined by

$$\langle f(x) \rangle = \sum_x f(x) P(x), \tag{1}$$

where $P(x)$ is the probability function.

For a single continuous variable it is defined by,

$$\langle f(x) \rangle = \int f(x) P(x) \, d\, x. \tag{2}$$

The expectation value satisfies

$$\langle a\, x + b\, y \rangle = a \langle x \rangle + b \langle y \rangle \tag{3}$$

$$\langle a \rangle = a \tag{4}$$

$$\left\langle \sum x \right\rangle = \sum \langle x \rangle. \tag{5}$$

Expected word length

$$
\begin{aligned}
E(LENGTH) &= 3 \times 0.068 \\
&+2 \times 0.035 \\
&+3 \times 0.0284 \\
&+2 \times 0.0257 \\
&+1 \times 0.0229 \\
&\vdots
\end{aligned}
$$

Words listed by frequency: the first 2000 most frequent words from the Brown Corpus (1,015,945 words)

	Word	Instances	% Frequency
1.	The	69970	6.8872
2.	of	36410	3.5839
3.	and	28854	2.8401
4.	to	26154	2.5744
5.	a	23363	2.2996
6.	in	21345	2.1010
7.	that	10594	1.0428
8.	is	10102	0.9943
9.	was	9815	0.9661
10.	He	9542	0.9392
11.	for	9489	0.9340
12.	it	8760	0.8623
13.	with	7290	0.7176
14.	as	7251	0.7137
15.	his	6996	0.6886
16.	on	6742	0.6636
17.	be	6376	0.6276
18.	at	5377	0.5293
19.	by	5307	0.5224
20.	I	5180	0.5099

Linearity of Expectation

Linear Operator

An operator $\tilde{L}$ is said to be linear if, for every pair of functions f and g and scalar t,

$$\tilde{L}(f+g)=\tilde{L}f+\tilde{L}g$$

and

$$\tilde{L}(tf)=t\tilde{L}f.$$

SOME THEOREMS ON EXPECTATION

Theorem 3-1: If c is any constant, then

$$E(cX) = cE(X) \tag{8}$$

Theorem 3-2: If X and Y are any random variables, then

$$E(X + Y) = E(X) + E(Y) \tag{9}$$

Theorem 3-3: If X and Y are independent random variables, then

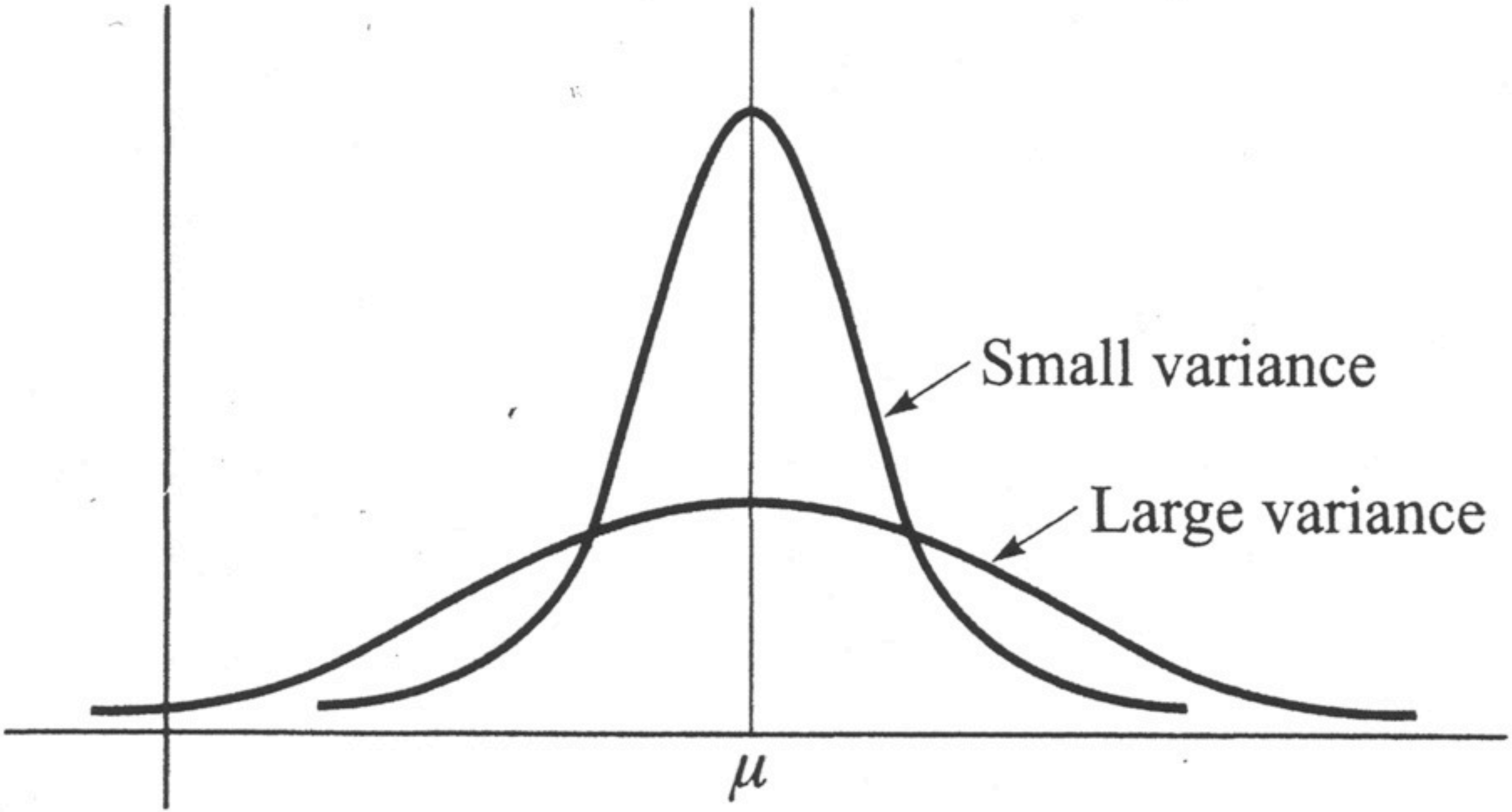
$$E(XY) = E(X)E(Y) \tag{10}$$

First die	Second die											
die	1	2	3	4	5	6						
6	7	8	9	10	11	12						
5	6	7	8	9	10	11						
4	5	6	7	8	9	10						
3	4	5	6	7	8	9						
2	3	4	5	6	7	8						
1	2	3	4	5	6	7						
x		2	3	4	5	6	7	8	9	10	11	12
$p(X = x)$		$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Figure 2.2 A random variable X for the sum of two dice. Entries in the body of the table show the value of X given the underlying basic outcomes, while the bottom two rows show the pmf $p(x)$.

cf. Johnson page 39

Variance, the expected squared deviation



Expected squared deviation

.... is just $E(X^2)$ - $(E(X))^2$

$$(130) \qquad \mathsf{V}(X) = \mathsf{E}(X^2) - (\mathsf{E}(X))^2$$

For a proof notice that

$$\begin{aligned}
\mathsf{E}(X - \mathsf{E}\,X)^2 &= \mathsf{E}(X - \mathsf{E}\,X)(X - \mathsf{E}\,X) \\
&= \mathsf{E}(X^2 - 2X \cdot \mathsf{E}\,X + (\mathsf{E}\,X)^2) \\
(131) \qquad &= \mathsf{E}(X^2) - 2\,\mathsf{E}((\mathsf{E}\,X) \cdot X) + (\mathsf{E}\,X)^2 \\
&= \mathsf{E}(X^2) - 2(\mathsf{E}\,X)(\mathsf{E}\,X) + (\mathsf{E}\,X)^2 \\
&= \mathsf{E}(X^2) - (\mathsf{E}\,X)^2
\end{aligned}$$

Expected squared deviation

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(130) $$V(X) = E(X^2) - (E(X))^2$$

expected squared deviation

Vasishth, appendix 4

For a proof notice that

$$E(X - E\,X)^2 = E(X - E\,X)(X - E\,X)$$
$$= E(X^2 - 2X \cdot E\,X + (E\,X)^2)$$
(131) $$= E(X^2) - 2\,E((E\,X) \cdot X) + (E\,X)^2$$
$$= E(X^2) - 2(E\,X)(E\,X) + (E\,X)^2$$
$$= E(X^2) - (E\,X)^2$$

SOME THEOREMS ON VARIANCE

Theorem 3-4: $$\sigma^2 = E[(X-\mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2 \tag{16}$$

where $\mu = E(X)$.

Theorem 3-5: If c is any constant,

$$\text{Var}\,(cX) = c^2\,\text{Var}\,(X) \tag{17}$$

Theorem 3-6: The quantity $E[(X-a)^2]$ is a minimum when $a = \mu = E(X)$.

Theorem 3-7: If X and Y are independent random variables,

$$\text{Var}\,(X+Y) = \text{Var}\,(X) + \text{Var}\,(Y) \quad \text{or} \quad \sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y \tag{18}$$

$$\text{Var}\,(X-Y) = \text{Var}\,(X) + \text{Var}\,(Y) \quad \text{or} \quad \sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y \tag{19}$$

SOME THEOREMS ON VARIANCE

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$$\sigma^2 = E[(X-\mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2 \quad (16)$$

where $\mu = E(X)$.

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Theorem 3-6: The quantity $E[(X-a)^2]$ is a minimum when $a = \mu = E(X)$.

Vasishth 2.4.1

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Sample of size 5

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average bitterness

$$\bar{X} = \frac{1}{5}\sum_{i=1}^{5} x_i$$

***n*=5**

⋮

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$$\vdots \qquad \vdots$$

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$$\vdots$$

the sampling distribution of the sample mean