February 15th
Expectation
Expectation

Suppose that a doctor get only two kinds of patients, those with insurance type `a’ and one with insurance `b’. If a patient has insurance `a’ the doctor gets 40 dollars per visit, if the patient is from insurance `b’ he gets 55. Let INCOME be a random variable from the event space of insurance types to the space of dollars defined by $f(A) = 40$ and $f(B) = 55$. $P(INCOME=a)$ is 1/3 and $P(INCOME=b)$ is 2/3. How much money does the doctor get on average from every patient? The answer is

$$1/3 \times 40 + 2/3 \times 55 = 50$$

This value is known as the expected value or expectation of INCOME.
MEAN: the value of a random variable that you expect, on average

$$
\mu_X = \sum_{\text{all possible values of } X} (\text{value of the R.V. } X) \times P(X)
$$
MEAN: the value of a random variable that you expect, on average

\[ \mu_X = \sum_{\text{all possible values of } X} (\text{value of the R.V. } X) \times P(X) \]

\[ \mu = E(X) \]
Expectation Value

The expectation value of a function $f(x)$ in a variable $x$ is denoted $\langle f(x) \rangle$ or $E\{f(x)\}$. For a single discrete variable, it is defined by

$$\langle f(x) \rangle = \sum_x f(x) P(x),$$

(1)

where $P(x)$ is the probability function.

For a single continuous variable it is defined by,

$$\langle f(x) \rangle = \int f(x) P(x) \, dx.$$

(2)

The expectation value satisfies

$$\langle a \, x + b \, y \rangle = a \langle x \rangle + b \langle y \rangle$$

(3)

$$\langle a \rangle = a$$

(4)

$$\langle \sum x \rangle = \sum \langle x \rangle.$$

(5)
Expected word length

\[ E(LENGTH) = 3 \times 0.068 + 2 \times 0.035 + 3 \times 0.0284 + 2 \times 0.0257 + 1 \times 0.0229 + \ldots \]
Linearity of Expectation

Linear Operator

An operator $\tilde{L}$ is said to be linear if, for every pair of functions $f$ and $g$ and scalar $t$,

$$\tilde{L}(f + g) = \tilde{L}f + \tilde{L}g$$

and

$$\tilde{L}(tf) = t\tilde{L}f.$$
SOME THEOREMS ON EXPECTATION

Theorem 3-1: If $c$ is any constant, then

$$E(cX) = cE(X)$$ \hfill (8)

Theorem 3-2: If $X$ and $Y$ are any random variables, then

$$E(X + Y) = E(X) + E(Y)$$ \hfill (9)

Theorem 3-3: If $X$ and $Y$ are independent random variables, then

$$E(XY) = E(X)E(Y)$$ \hfill (10)
<table>
<thead>
<tr>
<th>First die</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(X = x)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

**Figure 2.2** A random variable $X$ for the sum of two dice. Entries in the body of the table show the value of $X$ given the underlying basic outcomes, while the bottom two rows show the pmf $p(x)$.

cf. Johnson page 39
Variance, the expected squared deviation
Expected squared deviation

.... is just $E(X^2) - (E(X))^2$

(130) \[ V(X) = E(X^2) - (E(X))^2 \]

For a proof notice that

\[
E(X - E X)^2 = E(X - E X)(X - E X)
= E(X^2 - 2X \cdot E X + (E X)^2)
= E(X^2) - 2E((E X) \cdot X) + (E X)^2
= E(X^2) - 2(E X)(E X) + (E X)^2
= E(X^2) - (E X)^2
\]
Expected squared deviation

.... is just $E(X^2) - (E(X))^2$

\begin{align}
V(X) &= E(X^2) - (E(X))^2 \\
\text{For a proof notice that} \quad E(X - E X)^2 &= E(X - E X)(X - E X) \\
&= E(X^2 - 2X \cdot E X + (E X)^2) \\
&= E(X^2) - 2 E((E X) \cdot X) + (E X)^2 \\
&= E(X^2) - 2(E X)(E X) + (E X)^2 \\
&= E(X^2) - (E X)^2
\end{align}
SOME THEOREMS ON VARIANCE

**Theorem 3-4:** \[ \sigma^2 = E[(X - \mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2 \] (16)

where \( \mu = E(X) \).

**Theorem 3-5:** If \( c \) is any constant,

\[ \text{Var} (cX) = c^2 \text{Var} (X) \] (17)

**Theorem 3-6:** The quantity \( E[(X - a)^2] \) is a minimum when \( a = \mu = E(X) \).

**Theorem 3-7:** If \( X \) and \( Y \) are independent random variables,

\[ \text{Var} (X + Y) = \text{Var} (X) + \text{Var} (Y) \quad \text{or} \quad \sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y \] (18)

\[ \text{Var} (X - Y) = \text{Var} (X) + \text{Var} (Y) \quad \text{or} \quad \sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y \] (19)
SOME THEOREMS ON VARIANCE

Theorem 3-4: \[ \sigma^2 = E[(X - \mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2 \] (16)
where \( \mu = E(X) \).

Theorem 3-5: If \( c \) is any constant,
\[ \text{Var}(cX) = c^2 \text{Var}(X) \] (17)

Theorem 3-6: The quantity \( E[(X - a)^2] \) is a minimum when \( a = \mu = E(X) \).

Theorem 3-7: If \( X \) and \( Y \) are independent random variables,
\[ \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \] or \[ \sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y \] (18)
\[ \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) \] or \[ \sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y \] (19)
Sample of size 5

\[ n=5 \]
Sample of size 5

$n=5$

average bitterness

$$\bar{X} = \frac{1}{5} \sum_{i=1}^{5} x_i$$

Monday, February 15, 2010
Sample of size 5

average bitterness

\[ \bar{X} = \frac{1}{5} \sum_{i=1}^{5} x_i \]

\[ = \bar{X} \]

\[ = \bar{X} \]

\[ = \bar{X} \]

\[ = \bar{X} \]

\[ \vdots \]

\[ \vdots \]
Sample of size 5

\[ n=5 \]

\[ \bar{X} = \frac{1}{5} \sum_{i=1}^{5} x_i \]

average bitterness

the sampling distribution of the sample mean

Monday, February 15, 2010