Chapter 5: Bank Risk

A. Sources of Bank Risk

Risk management can be said to be the core business of the corporate bank, and inadequate risk management threatens the solvency and existence of the bank. The range of different potential risks faced by banks is wide-ranging, from economy-wide macroeconomic, political and natural disaster risks to the failure of a single customer. All of these risks are substantially magnified by the bank's financial risk, the risk associated with being extremely leveraged with debt. We have discussed major sources of bank risk in previous chapters, we will repeat and add to the list of these sources here, all of which have resulted in numerous bank failures:

Risks Associated with Counterparty Default

Credit risk, discussed more extensively in Sections 5.B, 6.D and Chapter 8 as a primary risk to any bank, concerns the potential that a loan or other credit asset experiences a default, or failure that any debtor of the bank fails to fulfill one of its obligations. Credit risk can be exacerbated by moral hazard, discussed in detail in the following section.

Counterparty risk, described here in the context of a traded financial instrument, is the risk that a counterparty will fail to fulfill the terms of its obligation. Technically, counterparty risk can include credit risk, sovereign risk and settlement risk.

Sovereign and political risk, including the risk that a national government will default on its debts (sovereign risk) to the bank or interfere in the foreign operations of a bank can be or result in a type of credit risk.

Settlement risk, also known as delivery risk, is the potential that a counterparty will fail to deliver assets as per the terms of a contract. Settlement risk might be related to failure to deliver securities by a trade counterparty, exchange or clearing firm.

Risks Associated with Rate and Price Uncertainty

Interest rate risk, discussed extensively here in Sections 5.D through 5.I and 6.C and further in Chapter 8 as a primary risk to any bank, concerns the potential that a shift in interest rates will diminish the value of loans and other assets, or increase the payout or value associated with a liability, or otherwise impair the institution's or its clients' abilities to fulfill their obligations.

Market risk, discussed later in this chapter, concerns the price uncertainty at which an instrument can be liquidated in the market. Market risk implies price uncertainty. Even if the security will not be sold at the time, most banks are subject to "marking to the market" regulatory requirements. Marking to the market accounts for the fair value of an asset or liability based on its current market price, or, if unavailable, based on either similar instruments or an appropriate valuation model. Uncertainty or volatility on market prices can seriously impact a bank's balance sheet. Market risk can include interest rate risk, commodity prices (such as oil), off-balance sheet volatilities and exchange rate risk. Off-balance sheet risks can be of particular concern since the contingent liabilities associated with off-balance sheet items can often exceed the actual assets or liabilities of the bank by a factor of more than 10. We discussed many such off-balance sheet instruments in Chapter 6. Diversification can be key to managing market risk.

Risks Associated with Inadequate Balance Sheet Accounts

Liquidity risk concerns the potential for failing to maintain a sufficient level of cash, near cash and other short-term or liquid assets to fulfill the institution's obligations or day-to-day operations in the near future. Liquidity risk can occur from or be exacerbated by the bank's inability to find a market for selling assets or be forced to sell at "fire-sale" prices (liquidity risk with respect to individual assets or liabilities). Unanticipated deposit withdrawals or liability redemptions can create or exacerbate bank liquidity risk. Many banks and other financial institutions have found that even a strong equity position can prove inadequate for ensuring that the institution can recover from liquidity shortfalls. However, maturity transformation is a key economic function of banks; banks maintain profitability by maintaining a maturity gap, by funding long-term loans with short-term deposits. The risks associated with maturity gaps can be mitigated by maturity and duration matching as described in Sections 5.F-5.I and by short-term depositors and other funders rolling over their claims on the bank. Deposit insurance certainly reduces liquidity risk, though may increase insolvency risk through moral hazard or by encouraging risky behavior.

Insolvency risk is the potential that the bank may not have sufficient capital (equity) to fulfill its regulatory requirements in order to offset a sudden asset value decline or liability value increase. These risks are also associated with maturity gaps, and can be mitigated by maturity and duration matching as described in Sections 5.F-5.I and by short-term depositors and other funders rolling over their claims on the bank.

Other Risks and Risk Management

Operational risk is the potential for "loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk" (Basel Committee on Banking Supervision (2011)). Potential for fraud, technological and system failure, loss of data, process management, human error, natural or war-related disasters, governance failures and lawsuits are among the many types of operational risk. Employee negligence, aggressive employee targets and incentives and unauthorized employee rogue trading (e.g., Société Générale’s Jérôme Kerviel, Barings Bank's Nick Leeson) and insufficient or failed internal bank controls can combine for huge losses. Legal risks can include new legislation, court opinions and regulations as well as the violations of laws and regulations.

Reputational risk is important because banks are highly dependent on sound or even pristine reputations for every line of their business activities. Reputational risk concerns the potential for damage to the bank's reputation caused by client or public perception and adverse publicity. Such damage can impair the bank's ability to conduct its activities and can diminish its revenues, financial and social capital. Reputational damage can result from a variety of situations or perceived activities that might include professional or ethics lapses, client or employee mistreatment, conflicts of interest, privacy and security issues, etc.

Risk management concerns the ongoing identification of risks, measuring the exposures to these risks, appropriate reporting and monitoring of these risks and controlling and mitigating these risks and the bank's exposures to them. The various risks described above can be interrelated, such as the bank that mismatches the maturity gaps in its foreign assets and liabilities, also intensifying its currency risk. As we will discuss below, a variety of risk management techniques are available for most of these risks.

B. Credit Risk

Lending is a core business function of the typical corporate bank, and credit risk is a key source of risk associated with lending. *Credit risk* refers to the potential that a borrower or counterparty fails to perform on an obligation. Since a large proportion of bank assets are loans, a primary source of credit risk to most banks will be loans and associated defaults. However, as we will discuss in the next chapter, a variety of types of marketable securities and derivative contracts are also important sources of credit risk (e.g., also known in this context as counterparty risk). Nevertheless, certain types of marketable and derivative securities can be used to mitigate credit and other risks.

Banks will establish and maintain credit philosophies that reflect their own markets, regulatory environments, risk tolerances, competitive positions and histories. These philosophies need to be articulated in formal written loan policies that should be supported and communicated throughout the lending arms of the institution. These policies will set loan exposure limits, articulate the responsibilities of all lending authorities playing roles in the credit extension process, provide guidance for any exceptions to these policies that might arise over time and be reviewed and revised as per the bank's strategic direction. The lending and credit cultures of the bank will reflect these credit philosophies. Banks manage credit risks associated with the loan portfolios through their:

1. Loan application processes: By carefully screening and scrutinizing the creditworthiness of prospective borrowers, meticulously following policies set forth by appropriate lending committees
2. Loan origination processes: By maintaining strict controls over loan approval and disbursement processes
3. Loan servicing and monitoring: By continuously monitoring borrower maintenance of loan agreement terms and behavior throughout the life of the loan
4. Loan portfolio diversification: Because incidences of loan defaults are not perfectly correlated, banks can reduce their overall credit risk by appropriately diversifying their loan portfolios. In addition, many can securitize, offload and insure loans in securities and derivatives markets.

<u>Loan Application Processes</u>

As we emphasized earlier in the text, banks engage in intensive screening processes to obtain information on prospective borrowers. This screening process produces private information that is the key element of any relationship that emerges between the bank and its client.

Most banks will have technology in place to provide systematic and cost-effective mechanical screening of initial loan applications. Additional screening information is normally easily obtainable through credit references from credit reporting agencies such as Experian, Equifax, Standard & Poors, Dunn & Bradstreet, etc. Most larger loans will ultimately require borrowers to submit detailed financial records along with tax, audit, operational and other records, data and analyses. Larger banks will have the technology and expertise to efficiently and effectively analyze such details to support lending decisions. Credit scoring programs and algorithms are typically used by banks for more quantitatively-based credit decisions, and are often geared to estimate probabilities of default (See, for example, Saunders (2000)). Much larger loans will sometimes result in a syndication process that will generally be managed by the initiating bank.

Loan Origination Processes

Banks need to establish systematic procedures for loan approvals and disbursement processes. Obviously, such processes are necessary for outright fraud prevention, but also to ensure prudent credit extension. Loan contracts can be written to provide for substantial credit risk mitigation. For example, a loan contract that provides for loan security with valuable collateral provides significant protection against default. Loan guarantor support, including letters of credit and credit insurance where applicable, needs to be evaluated and can enhance loan value. Writing the contract that enhances the priority standing of the lender enhances the lender's position in the event of bankruptcy. Loan *covenants*, specific requirements or restrictions imposed on the borrower can both reduce the likelihood of default and the lender's recovery in the event of default. Covenants restricting dividend payments to shareholders, restrictions on additional borrowing without bank approval, requirements to maintain and insure assets, requirements to submit to regular audits are among the more common types of restrictive covenants imposed on borrowers.

A number of proprietary management information systems such as the Moody's RiskCalc package, Moody's KMV Credit Monitor and JP Morgan's CreditMetrics are widely available to banks, and are designed to provide quantitative tools for assessing credit risk, as screening mechanisms, ongoing loan monitoring processes and as early warning tools. Some packages can aid in stress testing and meeting other regulatory requirements. Such packages can assist banks at every stage of the lending process.

In addition, banks need to examine and evaluate their overall lending activities on an ongoing basis. Consider, for example, ratios of loan losses or non-performing loans to total loans. Excessive or possibly increasing values for either of these ratios might suggest revisions in lending policy. Such ratios can be created for particular classes or types of loans extended by the bank.

More subtle lending or statistical trends have the potential to indicate problems with the loan origination process. For example, significant loan growth over time or increasing loan approval rates over time might indicate strengthening bank business; rapid loan growth and loan approval rates might also indicate aggressive lending predicated on deteriorating credit standards.

Loan Servicing and Monitoring

Monitoring borrowers is a key activity of the corporate bank. Banks accomplish this by regularly or continuously updating their information on borrowers, by visiting client facilities, meeting with client officers, regularly examining firm accounting statements and operational data. Corporate banks establish relationships with their borrowers, who benefit from easier access to credit and other bank facilities.

Ongoing banking relationships set the stage for effective ongoing monitoring of corporate clients; banks monitor the performance of existing loans to borrowers while considering the extension of additional loans and services to those same clients. These monitoring activities enable the bank to evaluate the ongoing performance of their loan assets and to take corrective measures should borrower activities increase the risks of outstanding loans. Such relationships are often strong enough that corporate boards will often have membership representation from their most important bank. Obviously, bank representation on a client's board of directors improves bank monitoring and the flow of information between banks and their borrowers.

Loan Portfolio Diversification

Undue concentrations of loan portfolios to specific borrowing clients, single industries or sectors or narrow geographic regions subject the bank to increased risk of distress and failure. While it is important for banks to seek their client niches and specialties, loan portfolio focus increases the risk of lending bank distress and failure. Banks reduce overall credit risk by diversifying borrower-specific credit risk, to the extent that sound business practices and regulators permit, across clients, sectors and industries and geographic regions. In addition, interrelationships between bank credit risk and other risks (e.g., interest rate risk) can be an important consideration to most banks.

C. Other Types of Default Risk

Any party with an obligation to the bank can default or otherwise fail to deliver as agreed. We used credit risk above in the context of a lending relationship, and review other forms of default risk below.

Counterparty Risk

Counterparty Risk concerns the potential that a counterparty in a securities or other financial trading transaction will fail to fulfill the terms of its obligation. For example, if a bank were to arrange an interest rate swap transaction with a broker, the bank might lose its potential profits from a profitable contract if the broker were to fail prior to the settlement date, thereby rendering the swap agreement worthless. Counterparty risk might include the potential for a counterparty to default on a forward contract by failing to deliver cash or instruments as obligated, the potential that a letter of credit, loan guarantee or other third-party contract to a loan will fail to be honored as can the potential for a payment order (check) given to a counterparty's bank to fail to clear or execute. Counterparty risk can be particularly acute during periods with volatile markets, as was clear with the failure of Lehman Brothers during the 2008 financial crisis.

Securities exchanges with central clearing parties can provide protection against counterparty risk when the exchanges and central clearing parties are less likely to default than counterparties. Hence, as with settlement risk, counterparty risk can be mitigated by trading exchange-listed derivatives and instruments rather than contracting with individual institutions or trading over-the-counter instruments. When this is not possible or practical, it is important for banks to establish early warning systems for counterparties with obligations, and to create and maintain watch lists of potentially troubled counterparties, and ensure that these lists are distributed among all relevant employees, including risk management, treasury and legal offices. Maximum threshold obligation amounts need to be established for each counterparty with which the bank trades, just as the bank will establish credit limits.

Sovereign and Political Risk

Sovereign risk is the potential that a national government borrower will default on its debt. Obviously, for lenders to governments, sovereign risk is an important type of credit risk. In some instances, governments will simply repudiate their debt obligations. For example, China, Cuba and North Korea repudiated their debts during the Cold War, refusing to acknowledge their debt agreements. Lending banks often have little recourse in the case of debt repudiation by sovereign countries beyond simply refusing to extend additional credit. Other countries, including Mexico and Argentina during the 1970s oil crisis and Greece during the early 21st century (and 4 other times since 1826) have rescheduled or restructured their repayments after defaulting. Rescheduling debt payments or restructuring them (reducing or converting them to contingency-based payments) usually requires an arduous negotiating process.

Political risk, while not technically a form of default risk, concerns the potential that a national government will interfere in the operations of a bank. Such interference can extend to surprises in tax policy, changes in lending or labor regulations or a variety of specific currency (e.g., devaluations or restrictions on bans on repatriation) or banking regulations. Loan diversification can be very helpful in mitigating sovereign and political risk.

Settlement Risk

In the previous section, we characterized settlement risk as the potential for a counterparty to fail to deliver assets as per the terms of a contract. For example, a trade counterparty in a forward contract might simply be unable to deliver treasury bills as per the forward agreement, defaulting on its agreement with the bank and potentially leaving the bank unable to fulfill its own contractual obligations or meet regulatory capital or liquidity requirements. A similar failure by an exchange or clearing firm trading a futures contract could have a similar impact on the bank. A delay by as little as a single day or even a few hours can be consequential as such a failure, especially if large, can trigger reverberation or domino effects throughout an entire banking system.

Settlement risk might also be considered a form of counterparty or credit risk. Most exchanges, including futures, commodities and FX exchanges use clearing firms and clearing houses to deliver securities. Modern clearance operations, including implementation of novation and netting procedures can significantly reduce settlement risk. However, forward contracts tend to be individually arranged and traded bank-to-bank rather than in organized exchanges. Hence, when available, the bank might benefit from the security that clearing firms offer on futures exchanges relative to forward contracts. A custodian's failure to deliver assets held in trust can also be considered a form of settlement risk. Nonetheless, regulatory limits on credit exposure, margin requirements, third party guarantees and collateral (be aware of International Swaps and Derivatives Association and national regulatory requirements concerning the rules for collateral management) and can also mitigate settlement risks.

D. Interest Rates[1]

For calculation purposes, it is often easier to assume that all discount and interest rates are equal for all periods, along with all bond yields. This means that the *yield curve*, which depicts the yields to maturity of zero-coupon bonds with respect to their terms to maturity, is flat.[2] But, of course, such assumptions are not realistic. Furthermore, interest rates do change over time, sometimes very unpredictably, and long term rates frequently exceed short term rates.

In this chapter, we discuss yields that vary among debt instruments over terms to maturity, and express long term interest rates as functions of short term rates. We will distinguish

[1] See Teall (2018).
[2] A *zero coupon bond*, also known as a *pure discount bond* or *strip* makes no explicit interest payments, but is purchased at a discount from its face or maturity value. Its yield is expressed as a function of its maturity and current market values.

between yields or rates on instruments originating at time zero or now (*spot rates*, on instruments originating now) and yields or rates on instruments originating in the future (*forward rates*, on instruments to be originated in the future at rates locked in at time zero or now). More specifically, we will argue that long term interest rates are related to the geometric mean of a series of short term spot rates and forward rates. The compounding effect of interest rates leads to long term rates being calculated based on geometric rather than arithmetic means. More specifically, at least initially, we will suggest that the long term spot rate will be expressed as a geometric mean of short term spot and forward interest rates.

Present Value and Discount Rates

Suppose that we wish to analyze a bond (a CD or any other interest-bearing debt instrument) maturing in n periods with a *face value* (or principle amount) equal to F (principal) paying interest annually at a rate of c. The annual interest payment is rate c multiplied by face value F (or cF). These interest payments are made at the end of each year. In addition, the bond makes a single payment of F at time n. Using a standard *present value* model discounting cash flows at an annual rate equal to k, the bond is evaluated as follows:

(5.1)

$$PV = \sum_{t=1}^{n} \frac{cF}{(1+k)^t} + \frac{F}{(1+k)^n}$$

For example, let c equal .10, F equal \$1000, k equal .12 and n equal 2. The present value of this bond is \$966.20, computed as follows:

$$PV = \frac{100}{(1+.12)^1} + \frac{100}{(1+.12)^2} + \frac{1000}{(1+.12)^2} = 966.20$$

Bond Yields

Present Value is used to determine the economic worth of a bond or other debt instrument; the return of a bond measures the profit relative to the investment of a bond. There are several measures of debt instrument return including *yield to maturity* y:

(5.2)
$$PV = \sum_{t=1}^{n} \frac{cF}{(1+y)^t} + \frac{F}{(1+y)^n}$$

Yield to maturity is, obviously, the value for y that satisfies Equation 5.2 (or k in Equation 5.1). Usually, a solution must be obtained through an iterative process. The yield to maturity (or internal rate of return) for the instrument described above is 12%, computed as follows:

$$P_0 = 966.20 = \frac{100}{(1+.12)^1} + \frac{100}{(1+.12)^2} + \frac{1000}{(1+.12)^2}$$

Thus, yield to maturity can be interpreted as that discount rate which sets the purchase price of a bond equal to its present value.

The Term Structure of Interest Rates

The *Term Structure of Interest Rates* is concerned with how yields and interest rates vary with respect to dates of maturity. The *Pure Expectations Theory* states that long term spot rates can be explained as a geometric mean of short term spot and forward rates. Thus, the Pure Expectation Theory defines the relationship between long and short term interest rates as follows, where $r_{0,n}$ is the rate on an instrument originated at time 0 and repaid at time n, and $r_{t-1,t}$ is the is the rate on an instrument originated at time $t-1$ and repaid at time t:

$$(5.3) \qquad r_{0,n} = \sqrt[n]{\prod_{t=1}^{n}(1+r_{t-1,t})} - 1$$

Thus, the first subscript provides that origination date of the relevant spot or forward rate and the second date provides its maturity date.

Illustration: Term Structure

Consider an example where the one-year spot rate $r_{0,1}$ is 2%. Investors are expecting that the one-year spot rate one year from now will increase to 3%, meaning that the one-year forward rate $r_{1,2}$ on loans originated in one year is 3%. Further suppose that investors are expecting that the one year spot rate two years from now will increase to 5%; thus, the one-year forward rate $r_{2,3}$ on a loan originated in two years is 5%. Based on the pure expectations hypothesis, the three-year spot rate in this market is calculated as follows (figures are rounded):

$$r_{0,3} = \sqrt{\prod_{t=1}^{n}(1+r_{t-1,t})} - 1 = \sqrt[n]{(1+r_{0,1})(1+r_{1,2})(1+r_{2,3})} - 1$$
$$= \sqrt[3]{(1+.02)(1+.03)(1+.05)} - 1 = .03326$$

This implies that a riskless borrower could obtain a three-year loan at 3.326%. Similarly, the two-year spot rate is calculated as follows:

$$r_{0,2} = \sqrt{\prod_{t=1}^{n}(1+r_{t-1,t})} - 1 = \sqrt{(1+r_{0,1})(1+r_{1,2})} - 1 = \sqrt{(1+.02)(1+.03)} - 1 = .025$$

For valuation purposes, these spot rates can be used as discount rates for their corresponding periods. Notice that as spot rates rise, the value of debt instruments will fall. This effect is more dramatic with longer-term instruments.

Illustration: Pricing a Bond

In an arbitrage-free market, any other bond or other debr instruments with cash flows paid at the ends of some combination of years 1, 2 and 3 must have a market price that is consistent with these three spot rates. For example, a 3-year 4% coupon Bond 4, discounted with the spot rates determined above, can be valued as follows:

$$PV[B_4] = \frac{40}{(1+.02)^1} + \frac{40}{(1+.025)^2} + \frac{1040}{(1+.03326)^3} = 1020.06$$

Any other market price for Bond 4 will lead to an arbitrage opportunity.

Calculating Forward Rates

Prices and cash flows from any combination of three of these bonds above will be consistent with the following forward rates (figures are rounded):

$$r_{1,2} = \frac{(1+r_{0,2})^2}{1+r_{0,1}} - 1 = \frac{(1+.025)^2}{1.02} - 1 = .03$$

$$r_{2,3} = \frac{(1+r_{0,3})^3}{(1+r_{0,2})^2} - 1 = \frac{(1+.03326)^3}{1.025^2} - 1 = .05$$

$$r_{1,3} = \sqrt{\frac{(1+r_{0,3})^3}{(1+r_{0,1})}} - 1 = \sqrt{\frac{1.03326^3}{1.02}} - 1 = .04$$

Table 5.1 summarizes all of the relevant spot and forward rates in this market given our initial one-year forward rates of $r_{0,1}$ =.02, $r_{1,2}$ =.03 and $r_{2,3}$=.05.

		Maturity Date		
		1	**2**	**3**
	0|	0.02	0.024988	0.033258
Origination Date	**1|**	N/A	0.03	0.039952
	2|	N/A	N/A	0.05

Table 5.1: Spot and Forward Rates Illustration

E. Interest Rate Risk

In general, the three primary types of debt instrument risk faced by banks might be categorized as follows:

1. *Default or credit risk*: Borrowers from banks might not fulfill all of their obligations.
2. *Liquidity risk*: An efficient market for banks to sell securities or otherwise raise capital might not exist or might be impaired.
3. *Interest rate risk*: Market interest rate fluctuations affect values of existing term loans and other assets, affect liability values as well as bank earnings.

In addition, many debt contracts can be called (redeemed or repaid) prior to maturity by the borrower or issuer, subjecting the bank to *call risk* or prepayment risk.

On the asset side of the bank's balance sheet, U.S. Treasury issues are generally regarded as being practically free of default risk. Furthermore, there exists an active market for Treasury issues, particularly those maturing within a short period. Thus, Treasury issues are regarded as having minimal liquidity risk as well. However, all debt instruments are subject to interest rate risk. Longer-term instruments are subject to increased interest rate risk due to the increased periods that the yields on longer-term instruments are likely to differ from shorter-term instruments. We will discuss hedging this interest risk in the next section.

Sources of Interest Rate Risk

Our primary concern in this section is interest rate risk. Interest rate risk draws from several factors, including the following:[3]

- *Repricing risk*: related to the cash flow structures and repricing of the bank's assets and liabilities along with relevant off balance sheet items. This risk typically involves or is related to maturity gaps, differences in the maturity structures between assets and liabilities. We will discuss maturity gaps and stress tests to analyze them in later chapters and corrective measures later in this chapter.
- *Basis risk*: arising from instruments created or sold in different markets, such as different countries, involving different types of instruments (e.g., mortgages vs. CDs, LIBOR vs. SOFR rates) institutions or industries, etc., causing the instruments to reprice differently.
- *Yield curve risk*: arising from changes in the slope and the shape of yield curves (non-parallel yield curve shifts)
- *Option risk*: arising from options held by counterparties, including options to retire debt or CDs early or extend loans when interest rates shift.

The Evolution of Short-Term Rates[4]

As we discussed earlier in this chapter, the yield curve depicts varying spot rates over associated terms to maturity. Understanding the nature of the uncertainty that drives spot and forward rates, particularly short term rates is essential to understanding fixed income instruments.

The Merton Model

The Merton [1973] term structure model prices bonds based on the assumption that short-term interest rates (more precisely, instantaneous spot and forward rates r_t) are related to an arithmetic Brownian motion process Z_t:

(5.4) $$dr_t = \mu dt + \sigma dZ_t$$

Instantaneous rates r_t following arithmetic Brownian motion are normally distributed, and normal distributions are often very easy to work with. On the other hand, in this model, changes in interest rates are unrelated to historical or long-term mean rates. This means that directional moves for short term interest rates cannot be predicted based on available information, particularly when the drift μ is low compared to interest rate volatility σ. The range of potential interest rate changes is from an unreasonable negative infinity to positive infinity. Negative interest rates are very possible under Brownian motion, but may seem less likely (though not

[3] See Federal Deposit Insurance Corporation (2018).
[4] This subsection, intended as a very superficial introduction to term structure models can be skipped without loss of continuity in the chapter.

impossible) in practice.

The Vasicek Model

In more realistic scenarios, we might observe that when the short-term spot rate seems high, that is it exceeds the long-term mean rate ($r_0 > \bar{r}$), the drift in the short term rate might be expected to be negative so that the short-term rate drifts down towards the long-term mean rate $\bar{r}$. We might say that interest rates are currently high in this scenario, and we expect for them to drop towards the long-term mean $\bar{r}$. When the short-term rate r_0 is less than the long-term rate ($r_0 < \bar{r}$), the drift might be expected to be positive. Thus, the short-term rate has a tendency to revert to its long-term mean $\bar{r}$, whose value might be the value justified by economic fundamentals such as capital productivity, long-term monetary policy, etc.

Define the term $0 < \lambda < 1$ to be a constant that reflects the speed of the mean-reverting adjustment for the instantaneous rate r_t towards its constant long-term mean rate $\bar{r}$; that is, λ is the mean reversion factor, sometimes called a "pullback factor." This pullback factor is typically estimated or calibrated based on a statistical analysis of historical data. Let σdZ_t represent Brownian motion shocks or random disturbances to r_t. If volatility σ is assumed to be independent of the short-term rate (e.g., it is a constant), the following defines the Ornstein-Uhlenbeck mean reverting process, also known as a *Vasicek process* in fixed income analysis:

(5.5) $$dr_t = \lambda(\bar{r} - r_t)dt + \sigma dZ_t$$

The Ornstein-Uhlenbeck process is sometimes called an elastic random walk. The Ornstein-Uhlenbeck process has two components, the mean reversion component $\lambda(\bar{r} - r_t)$ and the Brownian motion component σdZ_t. The Brownian motion component is the disturbance factor that causes the short-rate r_t to diverge from the long-term mean rate $\bar{r}$. The mean reversion component draws the short term rate r_t back towards the long term mean rate $\bar{r}$. A higher value for λ implies a faster reversion ($\lambda < 1$) by the short-term rate r_t towards the long term mean rate $\bar{r}$.

If λ were zero, there would be no mean reversion and the process would be a Brownian motion. One drawback to the Vasicek interest rate model is that interest rate shifts have a normally distributed component, leading to the unfortunate result that it is possible for the interest rate to become negative. Obviously, this creates an arbitrage opportunity when cash is available for investors to hold. Figure 5.1 depicts a simulation of a Vasicek process over length of time 200, with $r_0 = .05$, and $\sigma = .02$ and $\lambda = .1$. Also notice on Figure 5.1 that the process can drop to zero and that large changes in interest rates away from the long-term mean $\bar{r}$ tend to lead to large changes in the rate back towards the long-term mean.

The Vasicek yield curve model has a number of desirable characteristics. The model captures the empirical tendency for interest rates to revert towards some sort of mean rate. The model is driven off short term interest rates, much as actual interest rates might be impacted by the Federal Funds rate, the "overnight" bank-to-bank controlled by the central bank (Fed). However, there are a number of problems with the Vasicek model in characterizing the behavior of the yield curve:

1. The Vasicek model is likely to apply only in reasonably "normal" scenarios. For example, in situations involving crises such as hyperinflation, mean reversion is not likely to characterize the behavior of interest rates.

2. Because it is based on Brownian motion, the Vasicek model does not allow for discreet jumps in the interest rate process.
3. The Vasicek model produces the result that all short- and long-term rates are perfectly correlated.
4. Related to the difficulty put forth just above, the Vasicek model assumes only a single underlying risk factor when, in fact, there is significant evidence that there may well be multiple factors. For example, sometimes the yield curve can "twist;" that is, long- and short-term rates can move in opposite directions. Multiple risk factors can often explain such "twisting."
5. Finally, the Vasicek model allows for the possibility of negative interest rates, even for negative real interest rates, a phenomenon that we should expect to observe rarely, if at all.

Why work with an interest rate model that presents all of these difficulties? As with most other financial models, we simply balance realism and ease of model building. The Vasicek model does capture some of the characteristics of a reasonable interest rate process and it is rather easy to work with, particularly in terms of parameter calibration. In addition, it is useful and sometimes very straightforward to adapt this framework into more realistic alternative depictions of interest rate processes.

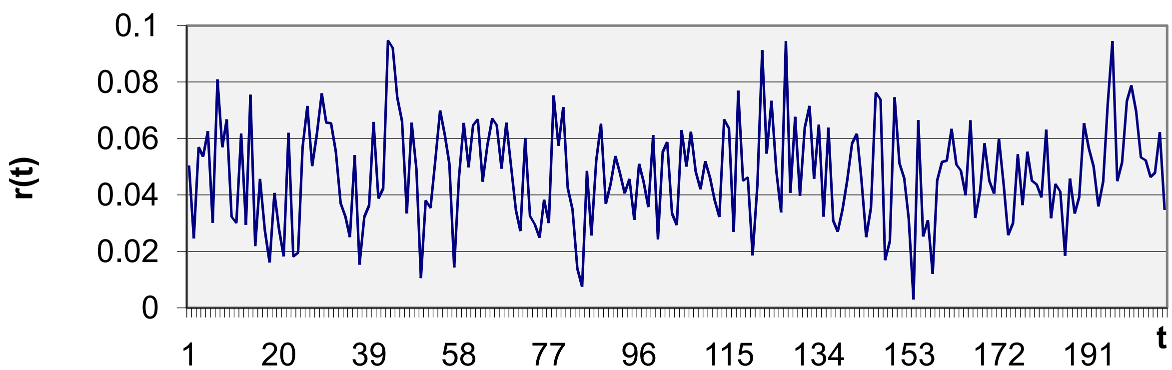

Figure 5.1: Simulation of Vasicek Process: r_0 = .05, σ = .02 and λ = .1

Bond Prices and Interest Rate Shifts

Equation 5.6 depicts a very simple relationship between interest rate shifts and a bond's discount rate or yield to maturity. As interest rates change in one direction, bond prices will change in the opposite direction. The longer one waits to recapture the value associated with his bond (the longer the bond's term to maturity), the greater will be the sensitivity of the bond's price to interest rate changes:

$$(5.6) \qquad PV = \sum_{t=1}^{n} \frac{cF}{(1+y)^t} + \frac{F}{(1+y)^n} = \sum_{t=1}^{n} cF(1+y)^{-t} + F(1+y)^{-n}$$

Let us consider how a change in interest rates might affect the value of a bond. Start by assuming that the terms of the bond contract, n, F and c are constant. To estimate this relationship, we will find the derivative of PV with respect to $(1+y)$:

(5.7) $$\frac{dPV}{d(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1}$$

Notice first that this derivative is negative, implying that interest rates and bond prices change in opposite directions. This relationship is depicted in Figure 5.2 for a zero coupon bond paying \$100 in 10 years. Notice in Figure 5.2 and Equations 5.6 and 5.7 that an increase in interest rates from, say 5% to 10% would significantly decrease the value of bank assets. Second, notice from Equations 5.6 and 5.7 that as *n*, the bond's term to maturity increases, this inverse relationship becomes stronger. This is a major problem for banks, which are so heavily engaged in the process of maturity transformation. Banks with short-term deposits and long-term financial assets face significant risks that result from even small changes in interest rates.

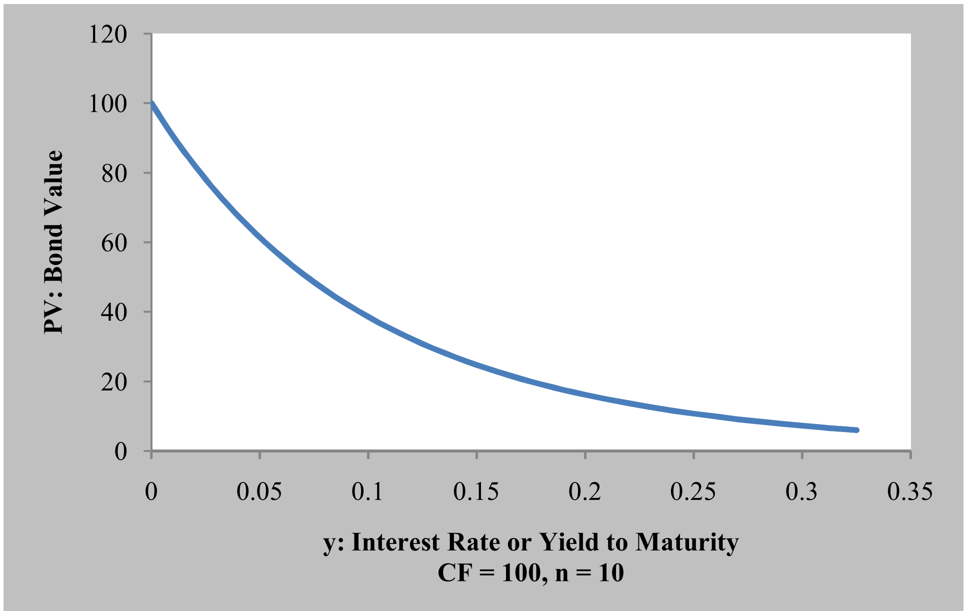

Figure 5.2: Zero Coupon Bond Price and Interest Rates

F. Asset-Liability Management

Asset-liability management, the coordinated management of accounts on both sides of the bank's balance sheet, might be the most important financial activity of the modern commercial bank. Major decisions concerning the management of interest rate risk are typically overseen by the bank's board of directors and made by a senior executive team often known as the *Asset-Liability Management Committee* (ALCO, sometimes called the Finance Committee), often including the bank CEO, COO, CFO, certain board members and other officers. The ALCO will establish and maintain interest rate risk measurement, monitoring and reporting systems, devise risk management strategies, institute internal control systems, impose position and risk limits and authorize policy exceptions. We will discuss many of these interest rate risk management functions in this and in the next two chapters. The ALCO will typically delegate its day-to-day operating responsibilities to the bank's treasury unit or investment officer.

<u>The Maturity Gap</u>

The modern corporate bank's emphasis on asset transformation, particularly maturity transformation makes bank profitability and total equity levels very sensitive to interest rate shifts. Demand deposits, a primary source of funding for most banks, are essentially money market instruments with zero terms to maturity. However, corporate bank borrowers tend to prefer take on longer-term debt, meaning that banks tend to lend from very short-term deposits. This mismatch in maturities is the basis of the *maturity gap*, which is sometimes defined as the weighted-average time to maturity of financial assets minus the weighted-average time to maturity of liabilities. While there are a number of variations on a standard gap ratio, one simple variation that appears in analyses is:

$$\text{Gap Ratio} = \frac{\text{Risk Sensitive Assets} - \text{Risk Sensitive Liabilities}}{\text{Average Earning Assets}}$$

Assets might be weighted by terms to maturity for another perspective.

	<1 Month	1-3 Months	3-12 Months	1-3 Years	3-10 Years	>10 Years	Total
Investments	20	5	5	0	0	0	30
Loans	10	5	5	20	20	10	70
Total Assets	30	10	10	20	20	10	100
Demand Deposits	-60	0	0	0	0	0	-60
CDs	-3	-4	-6	-4	-2	-1	-20
Subordinated Loans	-2	-1	-2	-2	-2	-1	-10
Total Liabilities	-65	-5	-8	-6	-4	-2	-90
Equity							-10
Periodic Gap	-35	5	2	14	16	8	0
Cumulative Gap	-35	-30	-28	-14	2	10	0

Table 5.2: Sample Abbreviated Gap Report for a Bank ($millions)

A *Gap report* is prepared by a bank to assess its repricing imbalances and interest rate exposure. A Gap report categorizes a bank's assets, liabilities and off-balance sheet instruments into maturity bands, summarizing asset-liability differences at each maturity band. For example, a bank might segment all of its assets and liabilities as depicted in Table 5.2. This bank is said to have a negative gap, since there are significant negative periodic and cumulative gaps in the shorter maturity terms in the table. These gaps lessen as the terms to maturity increase. This suggests that if interest rates were to rise, the decline in asset value would more than offset the decline in liability value (changes in asset and liability value are not depicted in the table), bank income would decline as would the value of the bank's equity.

Managing the Maturity Gap

Time deposits and fixed income instruments provide for fixed interest payments at fixed intervals and principal or balance repayments. In the absence of default and liquidity risk (and hybrid or adjustable features), uncertainties in interest rate shifts are the primary source of pricing risk for most fixed income instruments. In remainder of this chapter, we will discuss the analysis of fixed income instruments and how to manage interest rate uncertainty. In the next two chapters, we will introduce various types of securities used by banks to manage various risks and examples involving stress testing and asset-liability management. However, the following is a brief list of actions that a bank can undertake to manage its interest rate risk:

- Refuse to make intermediate- and long-term loans when the asset-liability maturity mismatch is too worrisome. Many banks will devise loan policies to prevent making loans large enough to create such mismatches. Unfortunately, banks can lose clients and miss out on profitable opportunities with this strategy.
- Seek to create assets with floating rate structures: While borrowers often prefer fixed-rate loans, banks will often seek to encourage borrowers to accept floating rate loans. In many instances, banks will need to offer very low interest rates on floating rate loans to entice borrowers to accept them.
- Seek to zero-out maturity gaps in the Gap report: This might be a useful and relatively simple way to deal with maturity gaps. However, the maturity segments can have fairly wide ranges, and any variation of ranges within a segment can lead to errors in the net effects of shifting interest rates.
- Cash flow dedication: Banks can seek to match exactly asset payoff structures to liability payoff structures so as to minimize the effects of interest rate shifts on bank income. We will discuss this strategy in the next section. Sometimes, finding assets or liabilities to complete the exact match is difficult or requires compromises in pricing or interest rate terms.
- Immunization: Banks can seek to match the interest rate sensitivity of asset values to the interest rate sensitivity of liability values. We will discuss this strategy later in this chapter.
- Employ derivative contracts: Forward rate agreements, interest rate swaps, futures contracts, floors, caps, collars and swaptions are among the instruments that can be used to help banks manage interest rate risk. Such techniques can create unintentionally complex risk exposures. We will briefly discuss such tools in the next chapter.
- Sell (offload) assets to move them off the balance sheets to reduce the bank's interest rate sensitivity. We will discuss this strategy in the next chapter.

G. Matching Asset and Liability Cash Flows

Banks seek to ensure relatively stable income levels over time and modest variability in equity capital. Typically, a bank must provide payments to its depositors for any given period and will collect payments on its loans and other assets over the same period. Accordingly, banks should invest in assets to ensure that depositor obligations and other liabilities are paid as needed. In many cases, banks will purchase assets to ensure that the cash flows produced by these assets exactly match the liability payments that the bank is required to make. This exact matching strategy is sometimes referred to as *dedication* and is intended to minimize the cash flow volatility or risk of the bank. Essentially, the bank manager determines the cash flows associated with the bank's liability (or asset) structure and replicates them with a series of assets (or liabilities). With cash inflows exactly matching cash outflows (along with profits), the bank's risk is minimized. Maturity mismatches between a bank's cash inflows and cash outflows are often revealed in the bank's gap report. Afterwards, the bank's treasury unit may undertake portfolio repositioning efforts to close this gap.

Dedication Illustration

Consider, for example, a bank that funds its operations with fairly short-term time deposits. Suppose that this bank has just extended balloon payment loans to three clients that are expected to produce payments to the bank of $12,000,000 in one year, $14,000,000 in two years, and $15,000,000 in three years. To hedge against interest rate increases that would devalue the loans, the bank will seek to fund these loans through a series of one, two and three year CD accounts. The bank's Asset-Liability Committee has decided to use a CD broker service to market these certificates of deposit to the general public.

Suppose that this bank has determined that it will sell one, two and three year CDs to match the cash flow structure of this loan portfolio with spot interest rates consistent with rates prevailing in the marketplace. Interest payments will be made on each of the CDs at the end of each of the three years until they mature. CD_1 will be a 1-year, 3% certificate, CD_2 will be a 2-year, 4% certificate, and CD_3 will be a 3-year, 5% certificate, with these CD rates being consistent with spot rates prevailing in the market. The next issue is to determine how many of each of the three certificates to sell to investors.

The cash flow structures of the bank's three loans will be matched to the cash flows associated with the bank's CD structure so as to eliminate variability in bank profits that might result from interest rate shifts. The bank will carry these loans on the asset side of its balance sheet and needs to issue CDs to investors that it will carry on the liabilities side of its balance sheet.

In year 1, the bank must pay its CD_1 holders $1030, CD_2 holders $40 and CD_3 holders $50. These payments should be combined to total $12,000,000, the cash inflow from loan recipients. Cash inflows must be matched to cash outflows in years 2 and 3 as well. Only one exact matching strategy will exist for this scenario, summarized by the equation system below. The following system can be solved for cd_1, cd_2 and cd_3 to determine exactly how many of each of the CDs should be sold to satisfy the bank's cash flow requirements:

$$
\begin{aligned}
12{,}000{,}000 &= 1030cd_1 + 40cd_2 + 50cd_3 \\
14{,}000{,}000 &= 1040cd_2 + 50cd_3 \\
15{,}000{,}000 &= 1050cd_3
\end{aligned}
$$

There are many ways to solve this system for the numbers the three CDs to dedicate to the bank's portfolio of assets. In this very simple case, we can *bootstrap* a solution, first solving the third equation for cd_3, then using that solution to solve the second equation for cd_2 and using the two solutions to solve the first equation for cd_1. Thus, we find from this system that our solutions are:

$$
\begin{aligned}
cd_1 &= 10{,}460.90 \\
cd_2 &= 12{,}774.73 \\
cd_3 &= 14{,}285.71
\end{aligned}
$$

Thus, with additional rounding, the bank should sell through its CD broker 10,461 CDs_1, 12,775 CDs_2, and 14.286 CDs_3. This set of CDs will exactly match the asset cash flow structure producing $12,000,000 in one year, $14,000,000 in two years, and $15,000,000 in three years.

The total purchase price associated with these CDs will be $37,521,337.89 (based on using spot rates for discounting and with minimal rounding). Interest rate shifts will not affect the cash flow structures of either assets or liabilities once they are locked in.

Issues with Dedication

Exact matching or dedication programs can be very effective when liquidity is sufficient to obtain deposits, loans and assets with appropriate cash flow structures when needed. However, what if, for example, a bank needs to lock in a cash flow from its portfolio of term loans equal to exactly $1,000,000 on a specific date exactly 12 years, 3 months and 2 days into the future? If no clients need to make term loans with this set of exact characteristics, some sort of approximation must be acceptable. Alternatively, what if such assets are available, but overpriced (or with unacceptably low interest rates)? Again, the manager must work with an approximation or accept investment in overpriced loan assets. In any case, dedication programs can limit managers with respect to what loans they can invest in and in what quantities they must granted. Next, we will prepare for a discussion on portfolio immunization, which is not always as effective as an interest rate hedging tool, but does allow for more flexibility.

H. Duration

As we discussed earlier, bonds and certain other debt instruments issued by the United States Treasury are often regarded to be practically free of default risk and of relatively low liquidity risk. However, these bonds, particularly those with longer terms to maturity are subject to market value fluctuations after they are issued, primarily due to changes in interest rates offered on new issues. Other bank assets will also be subject to interest rate risk. Generally, interest rate increases on new debt issues decrease values of bonds that are already outstanding; interest rate decreases on new debt issues increase values of bonds that are already outstanding. Immunization models such as the duration model are intended to describe the proportional change in the value of a bond induced by a small proportional change in interest rates or in yields of new issues.

Bond Duration

As discussed earlier, many analysts use present value models to value debt issues, frequently using yields to maturity of new issues as discount rates to value existing issues with comparable terms. It is important for analysts to know how changes in new-issue interest rates will affect values of bonds with which they are concerned. Bond *duration* measures the proportional sensitivity of a bond to changes in the market rate of interest.

Suppose that investors have valued a bond such that its market price equals its present value; that is, the discount rate k for the bond equals its yield to maturity y. If market interest rates and yields were rise for new treasury issues, then the yield of this bond would rise accordingly. However, since the contractual terms of the bond will not change, its market price must drop to accommodate a yield consistent with the market. Thus, increases in bond yields lead to bond price declines.

Deriving the Simple Macaulay Duration Formula

Assume that the value of an n-year bond paying coupon interest at a rate of c on face value F is determined by a present value model with the yield y of comparable issues serving as the discount rate k:

(5.2) $$PV = \sum_{t=1}^{n} \frac{cF_t}{(1+y)^t} + \frac{F}{(1+y)^n}$$

Assume that the terms of the bond contract, *n*, *F* and *c* are constant. Just what is the proportional change in the price of a bond induced by a proportional change in market interest rates (technically, a proportional change in [1+*y*])? This may be approximated by the bond's *Macaulay Simple Duration Formula* as follows:

(5.8) $$\frac{\Delta PV}{PV} \div \frac{\Delta(1+y)}{(1+y)} \approx Dur = \frac{dPV}{PV} \div \frac{d(1+y)}{(1+y)} = \frac{dPV}{d(1+y)} \times \frac{(1+y)}{PV}$$

Equation 5.8 provides a reasonable approximation of the proportional change in the price of a bond in a market meeting the assumptions described above induced by an infinitesimal proportional change in (1 + *y*). To derive this measure of a bond’s interest rate sensitivity (Equation 5.8), we first rewrite Equation 5.2 in polynomial form (to take derivatives later) and substitute *y* for *k* (since they are assumed to be equal):

(5.9) $$PV = \sum_{t=1}^{n} \frac{cF}{(1+y)^t} + \frac{F}{(1+y)^n} = \sum_{t=1}^{n} cF(1+y)^{-t} + F(1+y)^{-n}$$

First, find the derivative of PV with respect to (1+*y*):

(5.10) $$\frac{dPV}{d(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1}$$

Equation 5.11 can be rewritten:

(5.12) $$\frac{dPV}{d(1+y)} = \frac{\sum_{t=1}^{n} -tcF(1+y)^{-t} - nF(1+y)^{-n}}{(1+y)}$$

Since the market rate of interest is assumed to equal the bond yield to maturity, the bond's price P_0 will equal its present value *PV*. Next, multiply both sides of Equation 5.11 by (1+*y*)÷P_0 to obtain the bond’s proportional interest rate sensitivity, which is often more practical for portfolio purposes:

(5.13) $$Dur = \frac{dPV}{d(1+y)} \times \frac{(1+y)}{P_0} = \frac{\sum_{t=1}^{n} -tcF(1+y)^{-t} - nF(1+y)^{-n}}{P_0}$$

Equation 5.13 is equivalent to the right side of Equation 5.2. Thus, duration is defined as the proportional price change of a bond induced by an infinitesimal proportional change in (1+*y*) or

1 plus the market rate of interest:

$$(5.14) \qquad Dur = \frac{dPV}{d(1+y)} \times \frac{(1+y)}{P_0} = \frac{\sum_{t=1}^{n} \frac{-tcF}{(1+y)^t} + \frac{-nF}{(1+y)^n}}{P_0}$$

Illustration of Duration Calculation

Consider a two-year 10% coupon treasury issue which is currently selling for $966.20. The yield to maturity *y* of this bond is 12%. Default risk and liquidity risk are assumed to be zero; interest rate risk will be of primary importance. Assume that this bond's yield or discount rate is the same as the market yields of comparable treasury issues (which might be expected in an efficient market) and assume that bonds of all terms to maturity have the same yield. Further assume that investors have valued the bond such that its market price equals its present value; that is, the discount rate *k* for the bond equals its yield to maturity *y*.

Since the market rate of interest will likely determine the yield to maturity of any bond, the duration of the bond described above is determined as follows from Equation 9:

$$Dur = \frac{\frac{-1\cdot.1\cdot1000}{(1+.12)} + \frac{-2\cdot.1\cdot1000}{(1+.12)^2} + \frac{-2\cdot1000}{(1+.12)^2}}{966.20} = -1.907$$

This duration level of -1.907 suggests that the proportional decrease in the value of this bond will equal 1.907 times the proportional increase in market interest rates. This duration level also implies that this bond has exactly the same interest rate sensitivity as a *pure discount bond* (a bond making no coupon payments) that matures in 1.907 years. For example, if interest rates were to decline by .3% to 11.7%, the bond's price would rise by approximately (-1.97×-.3%×$966.20) = $5.71 to approximately $971.91 based on this duration calculation. Duration is considered a first order approximation (based on a first derivative only), hence its results should not be taken to be exact.

Application of the Simple Macaulay Duration model does require several important assumptions. First, it is assumed that yields are invariant with respect to maturities of bonds; that is, the yield curve is flat. Furthermore, it is assumed that investors' projected reinvestment rates are identical to the bond yields to maturity. Any change in interest rates will be infinitesimal and will also be invariant with respect to time. The accuracy of this model will depend on the extent to which these assumptions hold. We will discuss convexity later in the next section, which will allow for better estimates given larger changes in interest rates.

I. Immunization

Immunization is concerned with controlling asset and liability values in the face of interest rate uncertainty, seeking to ensure that institutional value is immunized from interest rate shifts.

Portfolio Immunization

Earlier, we discussed bond portfolio dedication, which minimizes interest rate risk by matching cash flows of bond portfolios with required payouts associated with liabilities. This process assumes that no transactions will take place within the portfolio and that cash flows associated with liabilities will remain as originally anticipated. Clearly, these assumptions will not hold for many institutions. Alternatively, one can hedge fixed income portfolio risk by using *immunization* strategies, which are concerned with matching the present values of asset portfolios with the present values of cash flows associated with future liabilities. More specifically, immunization strategies are primarily concerned with matching asset durations with liability durations. If asset and liability durations are matched, it is expected that the net fund value (equity or surplus) will not be affected by a very small shift in interest rates; asset and liability changes offset each other. Again, this simple immunization strategy is dependent on the following:

1. Changes in $(1 + y)$ are infinitesimal.
2. The yield curve is flat (yields do not vary over terms to maturity).
3. Yield curve shifts are parallel; that is, all short- and long-term interest rates change by the same amount and the yield curve maintains its shape.
4. Only interest rate risk is significant.

The first assumption, because it allows us to use calculus to measure sensitivities, can only be an approximation when interest rates change by finite amounts. We will discuss bond convexity shortly as a correction for this scenario. Assuming flat yield curves and parallel yield curve shifts are useful in that we do not have to distinguish between different rates (e.g., short- and long-term rates) over the term of the bond. Immunization becomes significantly more complicated when we need to analyze fixed income risks such as those related to liquidity and default.

Immunization Illustration

Consider a second illustration involving the bank in our dedication illustration above that funds its operations with fairly short-term time deposits. Continue to suppose that this bank has just extended balloon payment loans to three clients that are expected to produce payments to the bank of $12,000,000 in one year, $14,000,000 in two years, and $15,000,000 in three years. To hedge against interest rate increases that would devalue the loans, the bank will seek to fund these loans through a series of one, two and three year CD accounts. Again, the bank's Asset-Liability Committee has decided to use a CD broker service to market these certificates of deposit to the general public. Interest payments will be made on each of the CDs at the end of each of the three years until they mature. All of the CDs will be issued at coupon rates of 4%, consistent with the flat market yield curve, with CD_1 maturing in 1 year, CD_2 maturing in 2 years, and CD_3 maturing in 3 years. Our problem is to determine how many of each of the three certificates to sell to investors.

In this illustration, the pension fund manager still has anticipated cash payouts of $12,000,000, $14,000,000 and $15,000,000 over the next three years 1, 2 and 3. Now, suppose that the manager seeks to immunize interest rate risk associated with this liability stream by issuing CDs. Rather than exactly match the liability outflow streams with bond inflows, the manager will match durations of the CD liability stream with the duration of the loan investment portfolio. The managers will seek to ensure that changes in the value of the liability stream induced by interest rate changes is approximately the same as changes in the value of the loan portfolio. This will minimize fluctuations in the net value (assets minus liabilities) of the fund as interest rates vary. In addition, given the flat yield curve of 4%, the value of the portfolio of loans is $37,817,193.90.

We calculate bond and liability stream durations as follows:

$$\mathrm{Dur}_{\mathrm{CD1}} = \frac{\frac{1040}{1+0.04}}{-100} = -1$$

$$\mathrm{Dur}_{\mathrm{CD2}} = \frac{\frac{40}{1+0.04}+2\times\frac{1040}{(1+0.04)^3}}{-1000} = -1.962$$

$$\mathrm{Dur}_{\mathrm{CD2}} = \frac{\frac{40}{1+0.04}+2\times\frac{40}{(1+0.04)^2}+3\times\frac{1040}{(1+0.04)^3}}{-1000} = -2.886$$

$$\mathrm{Dur}_{\mathrm{L}} = \frac{\frac{12{,}000{,}000}{1+0.04}+2\times\frac{14{,}000{,}000}{(1+0.04)^2}+3\times\frac{15{,}000{,}000}{(1+0.04)^3}}{-37{,}817.193.90} = -2.048$$

Portfolio immunization is accomplished when the duration (weighted average duration) of the portfolio of loans equals the duration (-2.048) of the CDs issued by the bank:

$$\begin{aligned} \mathrm{Dur}_{\mathrm{CD1}} \cdot w_{\mathrm{CD1}} + \mathrm{Dur}_{\mathrm{CD2}} \cdot w_{\mathrm{CD2}} + \mathrm{Dur}_{\mathrm{CD3}} \cdot w_{\mathrm{CD3}} &= \mathrm{Dur}_{\mathrm{L}} \\ w_{\mathrm{CD1}} + w_{\mathrm{CD2}} + w_{\mathrm{CD3}} &= 1 \end{aligned}$$

$$\begin{aligned} -1 \cdot w_{\mathrm{CD1}} - 1.962 \cdot w_{\mathrm{CD2}} - 2.886 \cdot w_{\mathrm{CD3}} &= -2.048 \\ w_{\mathrm{CD1}} + w_{\mathrm{CD2}} + w_{\mathrm{CD3}} &= 1 \end{aligned}$$

There are an infinity of solutions to this two-equation, three variable system. Any solution that both satisfies these two equations and satisfies any other of the manager's other constraints and/or preferences is acceptable. For example, one solution to this system of equations results in an immunized portfolio with the following weights: $w_{CD1} = 0.1$, $w_{CD2} = .157836$ and $w_{CD3} = .742164$.

Duration immunized portfolios are most effective when interest rate changes are infinitesimal. Since interest rate changes are likely to be finite, and perhaps even large, immunization strategies will be improved if we correct for finite interest rate movements by using convexity. Duration is based on the first derivative of a bond's price with respect to interest rates. This first derivative, or first order approximation would be accurate only if the relationship were linear, which it is not. To correct for non-linearities in this relationship, we match asset and liability portfolio convexities as well as durations to correct for finite interest rate changes. We will discuss convexity calculations next.

Convexity[5]

In the previous section and subsection, we used duration to determine the approximate change in a bond's value induced by a change in interest rates $(1+y)$. However, the accuracy of the duration model is reduced by finite changes in interest rates, as we might expect. Duration may be regarded as a first order approximation (it only uses the first derivative) of the change in the value of a bond induced by a change in interest rates. *Convexity* is determined by the second

[5] This subsection can be skipped without loss of continuity in the chapter or remainder of the text.

derivative of the bond's value with respect to (1+y); that is, convexity is concerned with the change of the bond's value with respect to the change of the change in (1+y). Recall that the first derivative of the bond's price with respect to (1+y) is given:

$$(5.10) \qquad \frac{\partial P_0}{\partial(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1}$$

We find the second derivative by determining the derivative of the first derivative as follows:

$$(5.15) \qquad \begin{aligned} \frac{\partial^2 P_0}{\partial(1+y)^2} &= \left[\sum_{t=1}^{n} -t(-t-1)cF(1+y)^{-t-2}\right] - \left[n(-n-1)F(1+y)^{-n-2}\right] \\ &= \left[\sum_{t=1}^{n} \frac{(t^2+t)cF}{(1+y)^{t+2}}\right] + \left[\frac{(n^2+n)F}{(1+y)^{n+2}}\right] \end{aligned}$$

Convexity is merely the second derivative of P_0 with respect to (1+y) divided by P_0:

$$(5.16) \qquad Convexity = \frac{\left[\sum_{t=1}^{n} \frac{(t^2+t)cF}{(1+y)^{t+2}}\right] + \left[\frac{(n^2+n)F}{(1+y)^{n+2}}\right]}{P_0}$$

The first two derivatives can be used in a second order Taylor series expansion to approximate new bond prices induced by changes in interest rates as follows:

$$(5.17) \qquad P_1 \approx P_0 + f'(1+y_0)\cdot[\Delta(1+y)] + \frac{1}{2!}\cdot f''(1+y_0)\cdot[\Delta(1+y)]^2$$

$$(5.18) \qquad \begin{aligned} P_1 &\approx P_0 + \left[\sum_{t=1}^{n} \frac{-tcF}{(1+y_0)^{t+1}} - \frac{nF}{(1+y_0)^{n+1}}\right][\Delta y] \\ &+ \frac{1}{2}\left[\sum_{t=1}^{n} \frac{(t^2+t)\cdot cF}{(1+y_0)^{t+2}} + \frac{(n^2+n)\cdot F}{(1+y_0)^{n+2}}\right]\cdot[\Delta y]^2 \\ &= P_0 + Dur\cdot\tfrac{P_0}{1+y_0}\cdot[\Delta y] + \frac{1}{2}\cdot P_0\cdot convexity\cdot[\Delta y]^2 \end{aligned}$$

Convexity Illustration

Consider a 5-year ten-percent \$1000-face-value coupon bond currently selling at par (face value). We might compute the present yield to maturity of this bond as y_0 = .10. The first derivative of the bond's value with respect to (1+y) at y_0 = .10 is found from Equation 5.10 to be 3790.79 (duration is 3790.79 × 1.1 ÷ 1000 = 4.17); the second derivative is found from Equation 5.15 to be 19,368.34 (convexity is 19,368.34 ÷ 1000 = 19.37). If bond yields were to drop from .10 to .08, the actual value of this bond would increase to 1079.85, as determined from a standard present value model. If we were to use the duration model (first-order approximation from the Taylor expansion, based only on the first derivative), we estimate that the value of the bond increases to 1075.82. If we use the convexity model second-order approximation from Equation 16, we estimate that the value of the bond increases to 1079.69.

Note that this second estimate with the second-order approximation generates a revised bond value that is significantly closer to the bond's actual value as measured by the present value model. Therefore, the duration and immunization models are substantially improved by the second order approximations of bond prices (the convexity model). The fund manager wishing to hedge portfolio risk should not simply match durations (first derivatives) of assets and liabilities, he should also match their convexities (second derivatives).

Immunization Illustration

Now, let us reconsider our portfolio dedication illustration from Sections D and E along with the portfolio immunization illustration from above. In this illustration, the bank treasurer has anticipated cash payouts of \$12,000,000, \$14,000,000 and \$15,000,000 over the next three years. We calculate bond A, B and C convexities along with that for the liability stream as follows:

$$\mathrm{Conv}_A = \frac{2\times\frac{40}{(1+0.04)^3}+6\times\frac{1040}{(1+0.04)^4}}{1000} = 5.41$$

$$\mathrm{Conv}_B = \frac{2\times\frac{60}{(1+0.04)^3}+6\times\frac{60}{(1+0.04)^4}+12\times\frac{1060}{(1+0.04)^5}}{-10\quad .5} = 10.30$$

$$\mathrm{Conv}_C = \frac{12\times\frac{1000}{(1+0.04)^5}}{-88} = 11.09$$

$$\mathrm{Conv}_L = \frac{2\times\frac{12{,}000{,}000}{(1+0.04)^3}+6\times\frac{14{,}000{,}000}{(1+0.04)^4}+12\times\frac{15{,}000{,}000}{(1+0.04)^5}}{-37{,}816.120} = 6.38$$

Portfolio immunization is accomplished when the weighted averages of the duration and the convexity of the portfolio of bonds equals the duration and convexity (6.38) of the liability stream:

(5.19)
$$\begin{aligned} Dur_A \cdot w_A + Dur_B \cdot w_B + Dur_C \cdot w_C &= Dur_o \\ Conv_A \cdot w_A + Conv_B \cdot w_B + Conv_C \cdot w_C &= Conv_o \\ w_A + w_B + w_C &= 1 \end{aligned}$$

$$\begin{aligned} -1.962 \cdot w_A - 2.837 \cdot w_B - 3 \cdot w_C &= -1.975 \\ 5.41 \cdot w_A + 10.30 \cdot w_B + 11.09 \cdot w_C &= 6.38 \\ w_A + w_B + w_C &= 1 \end{aligned}$$

The single solution to this 3 X 3 system of equations is $w_A = 0.106$, $w_B = 5.166$ and $w_C = 4.272$. This system provides an improved immunization strategy over the duration benchmark alone.

Exercises

1. We did not use the term *refinancing risk* in this chapter. Typically, the term *refinancing* is used to describe the process of loan repayment, then obtaining a new loan, probably with different repayment terms. Based on this definition of refinancing, what type of risk discussed in this chapter would we most likely associate with refinancing risk?

2. A $1,000 face value bond is currently selling at a premium for $1,200. The coupon rate of this bond is 12% and it matures in three years. Calculate the following for this bond assuming its interest payments are made annually:

   a. Its annual interest payments.
   b. Its yield to maturity.

3. Work through each of the calculations in Problem 2 above assuming interest payments are made semi-annually.

4. Consider an example where we can borrow money today for one year at 5%; $y_{0,1} = .05$. Suppose that we are able to obtain a commitment to obtain a one year loan one year from now at an interest rate of 8%. Thus, the one year forward rate on a loan originated in year equals 8%. According to the Pure Expectations Theory, what is the two year spot rate of interest $y_{0,2}$?

5. Suppose that the one-year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one year spot rate one year from now will increase to 6%; thus, the one year forward rate $y_{1,2}$ on a loan originated in one year is 6%. Furthermore, assume that investors are expecting that the one year spot rate two years from now will increase to 7%; thus, the one year forward rate $y_{2,3}$ on a loan originated in two years is 7%. Based on the pure expectations hypothesis, what is the three-year spot rate?

6. Suppose that the one-year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one year spot rate one year from now will increase to 7%; thus, the one year forward rate $y_{1,2}$ on a loan originated in one year is 7%. Furthermore, assume that the three-year spot rate equals 7% as well. What is the anticipated one year forward rate $y_{2,3}$ on a loan originated in two years based on the pure expectations hypothesis?

7. Under what circumstances will borrowing short-term to fund long-term assets create bank losses? Under what circumstances will borrowing long-term to fund short-term assets create bank losses?

8. Banks engage in maturity transformation activities, maintain maturity gaps on their balance sheets and face risks because of these maturity gaps. Banks frequently have long-term assets funded with short-term liabilities. What types of institutions make natural trading partners for banks to offload their long-term assets?

9. A thrift institution expects to make payments of $30,000,000 in one year; $15,000,000 in two years; $25,000,000 in three years; and $35,000,000 in four years to its depositors. These

anticipated cash flows are to be matched with a portfolio of the following $1000 face value bonds:

BOND	CURRENT PRICE	COUPON RATE	YEARS TO MATURITY
1	1000	.10	1
2	980	.10	2
3	1000	.11	3
4	1000	.12	4

How many of each of the four bonds should the fund purchase to exactly match its anticipated payments to depositors?

10. Suppose that a thrift institution expects to make payments of $1,500,000 in one year; $2,500,000 in two years; and $4,000,000 in three years. These cash flows will be matched with a portfolio of bonds E, F and G whose characteristics are given in the table below. These three bonds must be used to match the cash flows associated with the fund's liability structure. For example, in year 1, Bond E will pay $1100, F will pay $120 and G will pay $100. These payments must be combined to total $1,500,000 for year 1. Cash flows must be matched in years 2 and 3 as well. How many of each of these bonds must be purchased or sold to exactly match the institution's cash flow needs? What will be the net investment into these bonds?

BOND	CURRENT PRICE	FACE VALUE	COUPON RATE	YEARS TO MATURITY
E	1010	1000	.10	1
F	1100	1000	.12	2
G	950	1000	.10	3

11. What is the duration of a 10% 10-year balloon payment loan?

12. While much of a bank's income from a typical loan derives from interest, banks sometimes require a lender to maintain a "*compensating balance*" in a no-interest or low-interest demand deposit account. How does the lending bank benefit from this practice?

Solutions

1. *Interest rate risk*: Since market rates of interest change nearly continuously, and such changes normally include a strong random element, predicting interest rates is very difficult. In addition, since most lenders express their reluctance to lend with higher interest rates, increased lender reluctance to lend increases interest rate risk.

2. a. Its annual interest payments:

$i_y = Int/F$

$Int = i_y(F)$; $= (.12)(1000)$ $= \$120$

b. Through substitution, we find yield to maturity to be .04697429 or 4.697429%

3. a. Its annual interest payments: \$120, or \$60 every six months.

b. Its yield to maturity *y* is found by substitution and eventually arriving at:

$0 = -1{,}200 + 60/[1+(y/2)]^1 + 60/[1+(y/2)]^2 + ...+ 60/[1+(y/2)]^5 + 1060/[1+(y/2)]^6$

$y = .0476634$

4. According to the Pure Expectations Theory, we compute the two year spot rate as follows:

$$(1+y_{0.2})^2 = \prod_{t=1}^{2}(1+y_{t-1,t}) = (1+.05)(1+.08) = 1.134$$

$$y_{0.2} = [(1+.05)(1+.08)]^{1/2} - 1 = \sqrt{1.134} - 1 = .0648944$$

5. The three-year rate is based on a geometric mean of the short term spot rates as follows:

$$(1+y_{0.3})^3 = \prod_{t=1}^{3}(1+y_{t-1,t}) = (1+.05)(1+.06)(1+.07) = 1.19091$$

$$y_{0.3} = [(1+.05)(1+.06)(1+.07)]^{1/3} - 1 = \sqrt[3]{1.19091} - 1 = .0599686$$

6. The three-year rate is based on a geometric mean of the short-term spot rates as follows:

$$(1+y_{0.3})^3 = (1.07)^3 = 1.22504 = \prod_{t=1}^{3}(1+y_{t-1,t}) = (1+.05)(1+.07)(1+y_{2,3})$$

We solve for $y_{2,3}$ as follows:

$$1.22504 \div [(1+.05)(1+.07)] - 1 = y_{2,3} = 0.0903$$

7. Insurance companies, particularly life insurance companies maintain long-term sources of funding from their policyholders. For example, life insurance policyholders frequently maintain their policies for decades. Insurance companies often prefer to build long-term asset portfolios in order to minimize their maturity gaps, and often make natural trading partners for banks.

8. Borrowing short-term to fund long-term assets will cause bank losses when interest rates rise. Borrowing long-term to fund short-term assets will cause bank losses when interest rates fall

9. The following matrix system may be solved for **b** to determine exactly how many of each of

the bonds are required to satisfy the fund's cash flow requirements:

$$\begin{bmatrix} 1100 & 100 & 110 & 120 \\ 0 & 1100 & 110 & 120 \\ 0 & 0 & 1110 & 120 \\ 0 & 0 & 0 & 1120 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 30,000,000 \\ 15,000,000 \\ 25,000,000 \\ 35,000,000 \end{bmatrix}$$

CF . b = Po

First, we invert Matrix **CF** to obtain $\mathbf{CF}^{-1}$:

$$\begin{bmatrix} .000909 & -.000083 & -.00008 & -.000079 \\ 0 & .000909 & -.00009 & -.000087 \\ 0 & 0 & .00090 & -.000096 \\ 0 & 0 & 0 & .000892 \end{bmatrix}$$

$\mathbf{CF}^{-1}$

Alternatively, without matrices, we set up and solve the following system:

$$1100b_1 + 100\,b_2 + 110\,b_3 + 120b_4 = 30000000$$
$$1100b_2 + 110\,b_3 + 120b_4 = 15000000$$
$$1110\,b_3 + 120b_4 = 25000000$$
$$1120b_4 = 35000000$$

First, solve the fourth equation, we obtain b_4:

$$b_4 = 31250$$

Now, substitute $b_4=31250$ in the original third equation:

$$1110\,b_3 + 120\times 31250 = 25000000$$

Solving this equation, we find that $b_3=19{,}144.14$. Now, substitute b_3, b_4 into the original second equation:

$$1100b_2 + 110\times 19{,}144.14 + 120\times 31250 = 15000000$$

Solve this equation, we know that $b_2=8{,}312.858$. Substitute b_2, b_3, b_4 into the original first equation:

$$1100b_1 + 100\times 8{,}312.858 + 110\times 19{,}144.14 + 120\times 31250 = 30000000$$

Finally, we find that $b_1=21{,}193.5$. Notice that we were able to use the bootstrapping method to solve his system. We find that the purchase of 21,193.5 Bonds 1, 8,312.858 Bonds 2, 19,144.14 Bonds 3 and 31,250 Bonds 4 satisfy the insurance company's exact matching requirements.

10. Only one matching strategy exists for this scenario. The following system may be solved for **b** to determine exactly how many of each of the bonds is required to satisfy the fund's cash flow requirements:

$$\begin{bmatrix} 1100 & 120 & 100 \\ 0 & 1120 & 100 \\ 0 & 0 & 1100 \end{bmatrix} \begin{bmatrix} b_E \\ b_F \\ b_G \end{bmatrix} = \begin{bmatrix} 1{,}500{,}000 \\ 2{,}500{,}000 \\ 4{,}000{,}000 \end{bmatrix}$$

$$\mathbf{CF} \times \mathbf{b} = \mathbf{L}$$

Inverting Matrix **CF** and multiplying by Vector **L**, we find that the purchase of 824.9704 Bonds E, 1907.467 Bonds F and 3636.363 Bonds G satisfy the manager's exact matching requirements. The fund's time zero payment for these bonds totals $6,385,979.9292.

11. 10 years: All payments on a balloon payment loan are made at maturity, which is 10 years for this balloon payment loan. The interest rate is irrelevant.

12. First, compensating balances, an indirect charge imposed on borrowers who are required to maintain deposit accounts, provide the bank with low-cost funding on which it can earn profits. Second, compensating balances can mitigate credit risk, serving as collateral in the event of borrower default.

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Appendix 5.A: Review of Matrices and Matrix Arithmetic

A *matrix* is simply an ordered rectangular array of numbers. A matrix is an entity that enables one to represent a series of numbers as a single object, thereby providing for convenient systematic methods for completing large numbers of repetitive computations. Such objects are essential for the management of large data structures. Rules of matrix arithmetic and other matrix operations are often similar to rules of ordinary arithmetic and other operations, but they are not always identical. In this text, matrices will usually be denoted with bold uppercase letters. When the matrix has only one row or one column, bold lowercase letters will be used for identification. The following are examples of matrices:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 7 & 4 \\ 8 & -5 & 9 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 2 & -3 \\ 3/4 & -1/2 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \mathbf{d} = [4]$$

The dimensions of a matrix are given by the ordered pair m × n, where m is the number of rows and *n* is the number of columns in the matrix. The matrix is said to be of *order m* × *n* where, by convention, the number of rows is listed first. Thus, **A** is 3 × 3, **B** is 2 × 2, **c** is 3 × 1, and **d** is 1 × 1. Each number in a matrix is referred to as an element. The symbol $a_{i,j}$ denotes the element in Row *i* and Column *j* of Matrix **A**, $b_{i,j}$ denotes the element in Row *i* and Column *j* of Matrix **B**, and so on. Thus, $a_{3,2}$ is -5 and $c_{2,1} = 5$.

There are specific terms denoting various types of matrices. Each of these particular types of matrices has useful applications and unique properties for working with. For example, a *vector* is a matrix with either only one row or one column. Thus, the dimensions of a vector are 1 × *n* or *m* × 1. Matrix **c** above is a column vector; it is of order 3 × 1. A 1 × *n* matrix is a row vector with *n* elements. The column vector has one column and the row vector has one row. A *scalar* is a 1×1matrix with exactly one entry, which means that a scalar is simply a number. Matrix **d** is a scalar, which we normally write as simply the number 4. A *square matrix* has the same number of rows and columns ($m = n$). Matrix **A** is square and of order 2. The set of elements extending from the upper- leftmost corner to the lower- rightmost corner in a square matrix are said to be on the *principal diagonal*. For each of these elements $i_{i,j}$, $i = j$. Principal diagonal elements of Square Matrix **A** are $a_{1,1} = 4$, $a_{2,2} = 7$ and $a_{3,3} = 9$. Matrices **B** and **d** are also square matrices.

A *symmetric matrix* is a square matrix where $c_{i,j}$ equals $c_{j,i}$ for all *i* and *j*; that is, the i^{th} element in each row equals the j^{th} element in each column. Scalar **d** and matrices **H**, **I**, and **J** below are all symmetric matrices. A *diagonal matrix* is a symmetric matrix whose elements off the principal diagonal are zero, where the *principal diagonal* contains the series of elements where $i = j$. Scalar **d** and Matrices **H**, and **I** below are all diagonal matrices. An *identity* or *unit* matrix is a diagonal matrix consisting of ones along the principal diagonal. Both matrices **H** and **I** following are diagonal matrices; **I** is the 3 × 3 identity matrix:

$$\mathbf{H} = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 10 \end{bmatrix} \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{J} = \begin{bmatrix} 1 & 7 & 2 \\ 7 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

<u>Matrix Arithmetic</u>

Matrix arithmetic provides for standard rules of operation just as conventional arithmetic. Matrices can be added or subtracted if their dimensions are identical. Matrices **A** and **B** add to **C** if $a_{i,j} + b_{i,j} = c_{i,j}$ for all i and j:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,n} \end{bmatrix}$$

A **B** **C**

Note that each of the three matrices is of dimension 3 × 3 and that each of the elements in Matrix **C** is the sum of corresponding elements in Matrices **A** and **B**. The process of subtracting matrices is similar, where $d_{i,j} - e_{i,j} = f_{i,j}$ for **D** - **E** = **F**:

$$\begin{bmatrix} d_{1,1} & d_{1,2} & \cdots & d_{1,n} \\ d_{2,1} & d_{2,2} & \cdots & d_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{m,1} & d_{m,2} & \cdots & d_{m,n} \end{bmatrix} - \begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ e_{m,1} & e_{m,2} & \cdots & e_{m,n} \end{bmatrix} = \begin{bmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{m,1} & f_{m,2} & \cdots & f_{m,n} \end{bmatrix}$$

D - **E** = **F**

Now consider a third matrix operation. The *transpose* $\mathbf{A}^T$ of Matrix **A** is obtained by interchanging the rows and columns of Matrix **A**. Each $a_{i,j}$ becomes $a_{j,i}$. The following represent Matrix **A** and its transpose $\mathbf{A}^T$:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}$$

A $\mathbf{A}^T$

Two matrices **A** and **B** can be multiplied to obtain the product **AB** = **C** if the number of columns in the first Matrix **A** equals the number of rows **B** in the second.[6] If Matrix **A** is of dimension $m \times n$ and Matrix **B** is of dimension $n \times q$, the dimensions of the product Matrix **C** will be $m \times q$. Each element $c_{i,k}$ of Matrix **C** is determined by the following sum:

$$c_{i,k} = \sum_{j=1}^{n} a_{i,j} b_{j,k}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \times \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,q} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,q} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,q} \end{bmatrix}$$

A × **B**

[6] If it is possible to multiply two matrices, they are said to be *conformable* for multiplication. Any matrix can be multiplied by a scalar, where the product is simply each element times the value of the scalar.

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1,j}b_{j,1} & \sum_{j=1}^{n} a_{1,j}b_{j,2} & \cdots & \sum_{j=1}^{n} a_{1,j}b_{j,q} \\ \sum_{j=1}^{n} a_{2,j}b_{j,1} & \sum_{j=1}^{n} a_{2,j}b_{j,2} & \cdots & \sum_{j=1}^{n} a_{2,j}b_{j,q} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{n} a_{m,j}b_{j,1} & \sum_{j=1}^{n} a_{m,j}b_{j,m} & \cdots & \sum_{j=1}^{n} a_{m,j}b_{j,q} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,q} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,q} \end{bmatrix}$$

A×B **C**

Notice that the number of columns (n) in Matrix **A** equals the number of rows in Matrix **B**. Also note that the number of rows in Matrix **C** equals the number of rows in Matrix **A**; the number of columns in **C** equals the number of columns in Matrix **B**.

An *inverse* Matrix $\mathbf{A}^{-1}$ exists for the square Matrix **A** if the products$\mathbf{AA}^{-1}$ or $\mathbf{A}^{-1}\mathbf{A}$ equal the identity Matrix **I**:

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$
$$\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$$

One means for finding the inverse Matrix $\mathbf{A}^{-1}$ for Matrix **A** is through the use of a process called the *Gauss-Jordan Method*.

Illustration: The Gauss-Jordan Method

An *inverse* Matrix $\mathbf{A}^{-1}$ exists for the square Matrix **A** if the product $\mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{AA}^{-1}$ equals the identity Matrix **I**. Consider the following product:

$$\begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{30} & \frac{2}{15} \\ \frac{4}{15} & \frac{-1}{15} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A $\mathbf{A}^{-1}$ **=** **I**

We will use the Gauss-Jordan Method to invert Matrix A by first augmenting **A** with the 2×2 identity matrix as follows:

(B) $$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 8 & 1 & 0 & 1 \end{array}\right]$$

For the sake of convenience, call the above augmented Matrix **B**. Now, a series of *elementary row operations* (addition, subtraction or multiplication of each element in a row) will be performed such that the identity matrix replaces the original Matrix **A** (on the left side). The right-side elements will comprise the inverse Matrix $\mathbf{A}^{-1}$. Thus, in our final augmented matrix, we will have ones along the principal diagonal on the left side and zeros elsewhere; the right side of the matrix will comprise the inverse of **A**. Allowable elementary row operations include the following:

1. Multiply a given row by any constant. Each element in the row must be multiplied by the same constant.
2. Add a given row to any other row in the matrix. Each element in a row is added to the corresponding element in the same column of another row.

3. Subtract a given row from any other row in the matrix. Each element in a row is subtracted from the corresponding element in the same column of another row.
4. Any combination of the above. For example, a row may be multiplied by a constant before it is subtracted from another row.

Our first row operation will serve to replace the upper left corner value with a one. We multiply Row 1 in **B** by .5:

$$\mathbf{B} = \begin{bmatrix} 2 & 4 & \vdots & 1 & 0 \\ 8 & 1 & \vdots & 0 & 1 \end{bmatrix} \overset{(row1)\times .5}{\rightarrow} \begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 8 & 1 & \vdots & 0 & 1 \end{bmatrix} = \mathbf{C}$$

Now we obtain a zero in the lower left corner by multiplying Row 2 in **C** by 1/8 and subtracting the result from Row 1 of **C** as follows:

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 8 & 1 & \vdots & 0 & 1 \end{bmatrix} \overset{row1-1/8(row2)}{\rightarrow} \begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 0 & \frac{15}{8} & \vdots & .5 & \frac{-1}{8} \end{bmatrix} = \mathbf{D}$$

Next, we obtain a 1 in the lower right corner of the left side of the matrix by multiplying Row 2 of matrix **D** by 8/15:

$$\mathbf{D} = \begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 0 & \frac{15}{8} & \vdots & .5 & \frac{-1}{8} \end{bmatrix} \overset{(row2)\times \frac{8}{15}}{\rightarrow} \begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 0 & 1 & \vdots & \frac{4}{15} & \frac{-1}{15} \end{bmatrix} = \mathbf{E}$$

We obtain a zero in the upper right corner of the left side matrix by multiplying Row 2 of matrix **E** above by 2 and subtracting from Row 1 in **E**:

$$\mathbf{E} = \begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 0 & 1 & \vdots & \frac{4}{15} & \frac{-1}{15} \end{bmatrix} \overset{row1-(row2)\times 2}{\rightarrow} \begin{bmatrix} 1 & 0 & \vdots & \frac{-1}{30} & \frac{2}{15} \\ 0 & 1 & \vdots & \frac{4}{15} & \frac{-1}{15} \end{bmatrix} = \mathbf{F}$$

The left side of augmented Matrix **F** is the identity matrix; the right side of **F** is $\mathbf{A}^{-1}$.

<u>Illustration: Solving Systems of Equations</u>

Matrices can be very useful in arranging systems of equations. Consider for example the following system of equations:

$$.05x_1 + .12x_2 = .05$$
$$.10x_1 + .30x_2 = .08$$

This system of equations can be represented as follows:

$$\begin{bmatrix} .05 & .12 \\ .10 & .30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .05 \\ .08 \end{bmatrix}$$

$$\mathbf{C} \quad \times \quad \mathbf{x} \quad = \quad \mathbf{s}$$

Thus, we can express this system of equations as the matrix equation **Cx**= **s**, where in general **C** is a given $n \times n$ matrix, **x** is a given $n \times 1$column vector, and **x** is the unknown $n \times 1$ column vector for which we wish to solve. In ordinary algebra, if we had the real-valued equation $Cx = s$, we would solve for s by dividing by both sides of the equation by A, which is equivalent to multiplying both sides of the equation by the inverse of A. Here we show the algebra, so that we see that this process with real numbers is essentially equivalent for the process with matrices:

$$Cx=s,\ C^{-1}Cx=C^{-1}s,\ 1(x)=C^{-1}s,\ x=C^{-1}s$$

With matrices, the process is:

$$\mathbf{Cx=s},\ \mathbf{C^{-1}Cx=C^{-1}s},\ \mathbf{Ix=C^{-1}s},\ \mathbf{x=C^{-1}s}.$$

Of course, in ordinary algebra, it is trivial to find the inverse of a number C, which is simply its reciprocal 1/C. To find the inverse of a matrix **C**, we use the Gauss-Jordan method described above. We begin by augmenting the matrix **C** by placing its corresponding identity matrix **I** immediately to its right:

(A) $$\left[\begin{array}{cc|cc} .05 & .12 & 1 & 0 \\ .10 & .30 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the allowable elementary row operations described earlier to the form with the identity matrix **I** on the left replacing **C**, and to the right will be the inverse of **C**:

(B) $$\left[\begin{array}{cc|cc} 1 & 2.4 & 20 & 0 \\ 0 & .6 & -20 & 10 \end{array}\right] \begin{array}{l} RowB1 = A1 \cdot 20 \\ RowB2 = (10 \cdot A2) - B1 \end{array}$$

(C) $$\left[\begin{array}{cc|cc} 1 & 0 & 100 & -40 \\ 0 & 1 & \frac{-100}{3} & \frac{50}{3} \end{array}\right] \begin{array}{l} RowC1 = B1 - (2.4 \cdot C2) \\ RowC2 = B2 \cdot 5/3 \end{array}$$

$$\mathbf{I} \qquad \mathbf{C^{-1}}$$

Thus, we obtain Vector **x** with the following product:

(D) $$\begin{bmatrix} 100 & -40 \\ \frac{-100}{3} & \frac{50}{3} \end{bmatrix} \begin{bmatrix} .05 \\ .08 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.8 \\ \frac{-1}{3} \end{bmatrix}$$

$$\mathbf{C^{-1}} \quad \mathbf{s} \quad = \quad \mathbf{x} \quad = \quad \mathbf{x}$$

Thus, we find that $x_1 = 1.8$ and $x_2 = -1/3$.

Appendix 4.A Exercises

1. Add the following matrices:

$$\underset{\mathbf{A}}{\begin{bmatrix} 2 & 4 & 9 \\ 6 & 4 & 25 \\ 0 & 2 & 11 \end{bmatrix}} + \underset{\mathbf{B}}{\begin{bmatrix} 3 & 0 & 6 \\ 2 & 1 & 3 \\ 7 & 0 & 4 \end{bmatrix}} =$$

2. Subtract **E** from **D**:

$$\underset{\mathbf{D}}{\begin{bmatrix} 9 & 4 & 9 \\ 6 & 4 & 8 \\ 5 & 2 & 9 \end{bmatrix}} - \underset{\mathbf{E}}{\begin{bmatrix} 5 & 0 & 6 \\ 2 & 1 & 6 \\ 5 & 0 & 9 \end{bmatrix}} =$$

3. Transpose the following:

a. $\underset{\mathbf{A}}{\begin{bmatrix} 1 & 8 & 9 \\ 6 & 4 & 25 \\ 3 & 2 & 35 \end{bmatrix}}$

b. $\underset{\mathbf{y}}{\begin{bmatrix} 9 \\ 6 \\ 3 \\ 7 \end{bmatrix}}$

c. $\underset{\mathbf{V}}{\begin{bmatrix} .09 & .01 & .04 \\ .01 & .16 & .10 \\ .04 & .10 & .64 \end{bmatrix}}$

4. Multiply the following:

$$\underset{\mathbf{A}}{\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix}} \times \underset{\mathbf{B}}{\begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix}}$$

5. Invert the following matrices:

a. $[8]$ b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ e. $\begin{bmatrix} .02 & .04 \\ .06 & .08 \end{bmatrix}$ f. $\begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$

g. $\begin{bmatrix} 33.33 & -8.33 \\ -8.33 & 8.33 \end{bmatrix}$ h. $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 20 \end{bmatrix}$

6. Solve each of the following for **x**:

a.
$$\underset{\boldsymbol{C}}{\begin{bmatrix} 33.33 & -8.33 \\ -8.33 & 8.33 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\boldsymbol{s}}{\begin{bmatrix} .01 \\ .11 \end{bmatrix}}$$

b.
$$\underset{\boldsymbol{C}}{\begin{bmatrix} .08 & .08 & .1 & 1 \\ .08 & .32 & .2 & 1 \\ .1 & .2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{\boldsymbol{s}}{\begin{bmatrix} .1 \\ .1 \\ .1 \\ .1 \end{bmatrix}}$$

Appendix 4.A Exercise Solutions

1. The sum is as follows:

$$\underset{\mathbf{A}}{\begin{bmatrix} 2 & 4 & 9 \\ 6 & 4 & 25 \\ 0 & 2 & 11 \end{bmatrix}} + \underset{\mathbf{B}}{\begin{bmatrix} 3 & 0 & 6 \\ 2 & 1 & 3 \\ 7 & 0 & 4 \end{bmatrix}} = \underset{\mathbf{C}}{\begin{bmatrix} 5 & 4 & 15 \\ 8 & 5 & 28 \\ 7 & 2 & 15 \end{bmatrix}}$$

2. The difference is as follows:

$$\underset{\mathbf{D}}{\begin{bmatrix} 9 & 4 & 9 \\ 6 & 4 & 8 \\ 5 & 2 & 9 \end{bmatrix}} - \underset{\mathbf{E}}{\begin{bmatrix} 5 & 0 & 6 \\ 2 & 1 & 6 \\ 5 & 0 & 9 \end{bmatrix}} = \underset{\mathbf{F}}{\begin{bmatrix} 4 & 4 & 3 \\ 4 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}}$$

3.a.
$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 8 & 9 \\ 6 & 4 & 25 \\ 3 & 2 & 35 \end{bmatrix}} \underset{\mathbf{A}^T}{\begin{bmatrix} 1 & 6 & 3 \\ 8 & 4 & 2 \\ 9 & 25 & 35 \end{bmatrix}}$$

b. The transpose of a column vector is a row vector:

$$\underset{\mathbf{y}}{\begin{bmatrix} 9 \\ 6 \\ 3 \\ 7 \end{bmatrix}} \underset{\mathbf{y}^T}{\begin{bmatrix} 9 & 6 & 3 & 7 \end{bmatrix}}$$

Similarly, the transpose of a row vector is a column vector.

c. Note that the transpose $\mathbf{V}^T$ of a symmetric matrix **V** is **V**:

$$\mathbf{V} = \begin{bmatrix} .09 & .01 & .04 \\ .01 & .16 & .10 \\ .04 & .10 & .64 \end{bmatrix} \mathbf{V}^T = \begin{bmatrix} .09 & .01 & .04 \\ .01 & .16 & .10 \\ .04 & .10 & .64 \end{bmatrix} = \mathbf{V}$$

4. Matrix **C**, the product of **A** and **B** is found as follows:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} (7\cdot7)+(4\cdot5)+(9\cdot9) & (7\cdot6)+(4\cdot1)+(9\cdot12) \\ (6\cdot7)+(4\cdot5)+(12\cdot9) & (6\cdot6)+(4\cdot1)+(12\cdot12) \\ (3\cdot7)+(2\cdot5)+(17\cdot9) & (3\cdot6)+(2\cdot1)+(17\cdot12) \end{bmatrix}$$

A × **B** = **C**

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 150 & 154 \\ 170 & 184 \\ 184 & 224 \end{bmatrix}$$

A × **B** = **C**

Notice that the number of columns (3) in Matrix **A** equals the number of rows in Matrix **B**. Also note that the number of rows in Matrix **C** equals the number of rows in Matrix **A**; the number of columns in **C** equals the number of columns in Matrix **B**.

5.a. 1/8 = .125

b. The inverse of the identity matrix is the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. The inverse of a diagonal matrix is found by inverting each of the principle diagonal elements:

$$\begin{bmatrix} .25 & 0 \\ 0 & 2 \end{bmatrix}$$

d. First, augment the matrix with the Identity Matrix:

$$\begin{matrix} row\ 1 \\ row\ 2 \end{matrix} \begin{bmatrix} 1 & 2 & \vdots & 1 & 0 \\ 3 & 4 & \vdots & 0 & 1 \end{bmatrix}$$

Now use the Gauss-Jordan Method to transform the original matrix to an identity matrix; the resulting right-hand side will be the inverse of the original matrix:

$$\begin{matrix} 1a \\ 2a \end{matrix} \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & -1 & \frac{1}{3} \end{bmatrix} \begin{matrix} row\ 1\ \times 1 \\ row\ 2\ \times \frac{1}{3} - (1a) \end{matrix}$$

$$\begin{matrix} 1b \\ 2b \end{matrix} \begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & 1.5 & -.5 \end{bmatrix} \begin{matrix} (1a) - 2\ \times (2b) \\ (2a) \times -1/.6 \end{matrix}$$

Thus, the inverse matrix is:

$$\begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$$

e.

$$\begin{bmatrix} .02 & .04 & | & 1 & 0 \\ .06 & .08 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 50 & 0 \\ 0 & \frac{2}{3} & | & 50 & -16\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -100 & 50 \\ 0 & 1 & | & 75 & -25 \end{bmatrix}$$

The inverse matrix is:

$$\begin{bmatrix} -100 & 50 \\ 75 & -25 \end{bmatrix}$$

f. The inverse matrix is: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

g. The inverse matrix is: $\begin{bmatrix} .04 & .04 \\ .04 & .16 \end{bmatrix}$

h. $$\begin{bmatrix} 2 & 0 & 0 & | & 1 & 0 & 0 \\ 2 & 4 & 0 & | & 0 & 1 & 0 \\ 4 & 8 & 20 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & .5 & 0 & 0 \\ 0 & 4 & 0 & | & -1 & 1 & 0 \\ 0 & 8 & 20 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & .5 & 0 & 0 \\ 0 & 1 & 0 & | & -.25 & .25 & 0 \\ 0 & 0 & 2 & | & 0 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & .5 & 0 & 0 \\ 0 & 1 & 0 & | & -.25 & .25 & 0 \\ 0 & 0 & 1 & | & 0 & -.1 & .05 \end{bmatrix}$$

The inverse matrix is:

$$\begin{bmatrix} .5 & 0 & 0 \\ -.25 & .25 & 0 \\ 0 & -.1 & .05 \end{bmatrix}$$

6.a. See 5.g above for the inverse of **C**:

$$C^{-1} = \begin{bmatrix} .04 & .04 \\ .04 & .16 \end{bmatrix}$$

$$\begin{bmatrix} .04 & .04 \\ .04 & .16 \end{bmatrix} \cdot \begin{bmatrix} .01 \\ .11 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .0048 \\ .018 \end{bmatrix}$$

$$\boldsymbol{C^{-1}} \cdot \mathbf{s} = \mathbf{x} = \mathbf{x}$$

b. Our original system of equations is represented:

$$\begin{bmatrix} .08 & .08 & .1 & 1 \\ .08 & .32 & .2 & 1 \\ .1 & .2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} .1 \\ .1 \\ .1 \\ .1 \end{bmatrix}$$

$$\boldsymbol{C} \cdot \mathbf{x} = \mathbf{s}$$

The elements of **C** and **s** are known; our problem is to find the weights in vector **x**. Thus we will rearrange the system from **Cx** = **s** to $\mathbf{C}^{-1}$s = **x**, where $\mathbf{C}^{-1}$ is the inverse of matrix **C**. So, the time-consuming part of our problem is to find $\mathbf{C}^{-1}$. We will begin by augmenting Matrix **C** with the Identity Matrix **I**:

$$\begin{matrix} Row\ 1 \\ Row\ 2 \\ Row\ 3 \\ Row\ 4 \end{matrix} \begin{bmatrix} .08 & .08 & .1 & 1 & \vdots & 1 & 0 & 0 & 0 \\ .08 & .32 & .2 & 1 & \vdots & 0 & 1 & 0 & 0 \\ .1 & .2 & 0 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} Original \\ System \end{matrix}$$

$$\begin{matrix} 1a \\ 2a \\ 3a \\ 4a \end{matrix} \begin{bmatrix} 1 & 1 & 1.25 & 12.5 & | & 12.5 & 0 & 0 & 0 \\ 0 & 3 & 1.25 & 0 & | & -12.5 & 12.5 & 0 & 0 \\ 0 & 1 & -1.25 & -12.5 & | & -12.5 & 0 & 10 & 0 \\ 0 & 0 & -1.25 & -12.5 & | & -12.5 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} (row1) \cdot 12.5 \\ (row2) \cdot 12.5 - (1a) \\ (row3) \cdot 10(1a) \\ (row4) \cdot 1 - (1a) \end{matrix}$$

$$\begin{matrix} 1c \\ 2c \\ 3c \\ 4c \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 6.25 & | & 12.5 & -6.25 & 5 & 0 \\ 0 & 1 & 0 & -3.125 & | & -6.25 & 3.125 & 2.5 & 0 \\ 0 & 0 & 1 & 7.5 & | & 5 & 2.5 & -6 & 0 \\ 0 & 0 & 0 & -3.125 & | & -6.25 & 3.125 & 7.5 & 1 \end{bmatrix} \begin{matrix} (1b)-(3c)\cdot .83 \\ (2b)-(3c)\cdot .416 \\ (3b)\cdot -1/1.6 \\ (4b)-(3c)\cdot -1.25 \end{matrix}$$

$$\begin{matrix} 1d \\ 2d \\ 3d \\ 4d \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & 0 & -10 & 2 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 10 & -1 \\ 0 & 0 & 1 & 0 & | & -10 & 10 & -24 & 2.4 \\ 0 & 0 & 0 & 1 & | & 2 & -1 & 2.4 & -.32 \end{bmatrix} \begin{matrix} (1c)-(4d)\cdot 6.2 \\ (2c)-(4d)\cdot -3.125 \\ (3c)-(4d)\cdot 7.5 \\ (4c)\cdot -1/(3.125) \end{matrix}$$

I *C*$^{-1}$

$$\begin{bmatrix} 0 & 0 & -10 & 2 \\ 0 & 0 & 10 & -1 \\ -10 & 10 & -24 & 2.4 \\ 2 & -1 & 2.4 & -.32 \end{bmatrix} \begin{bmatrix} .1 \\ .1 \\ .1 \\ .1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -.8 \\ .9 \\ -2.16 \\ .308 \end{bmatrix}$$

$$\boldsymbol{C}^{-1} \cdot \boldsymbol{s} = \boldsymbol{x} = \boldsymbol{x}$$

Now it is clear that:

x_1 = (0×.1) + (0×.1) + (-10×.1) + (2×.1) = -.8
x_2 = (0×.1) + (0×.1) + (10×.1) + (-1×.1) = .9
x_3 = (-10×.1) + (10×.1) + (-24×.1) + (2.4 × .1) = -2.16
x_4 = (2×.1) + (-1×.1) + (2.4×.1) + (-.32×.1) = .308

Appendix 5.B: Matrices and Spreadsheets

Solving equations with matrices of higher order by hand or with a calculator can be an extremely time-consuming and frustrating process. However, spreadsheets can be used quite effectively to multiply and invert matrices. Suppose that we wished to solve the following for **x** using an Excel™ spreadsheet:

$$\begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix},$$

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{s}.$$

We may start to solve this system by insert the coefficients from matrix **C** in the spreadsheet as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4					
5					

To solve the system, we will first invert matrix **C** in cells A1:B2. First, use the mouse to highlight cells A4 to B5, where we will insert the inverse of matrix **C**. We then select from the toolbar at the top of the screen the Paste Function button (f_x). From the Paste Function menu, we select the MATH & TRIG sub-menu. In the MATH & TRIG sub-menu, we scroll down to select MINVERSE, the function, which inverts the matrix. The MINVERSE function will prompt for an array; we enter the location of the matrix to be inverted: A1:B2. To fill all four cells A4 to B5, we simultaneously hit the Ctrl, Shift, and Enter keys. This is important. The spreadsheet should then appear as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4	0.15	−0.1			
5	−0.05	0.2			

The matrix in cells A4:B5 is $\mathbf{C}^{-1}$. Now, we enter into cells C4 and C5 the equation solution vector **s** containing elements 10 and 20. Then highlight cells D4 and D5 to solve for vector **x**, left click again the Paste Function key in the Toolbar and select the MATH & TRIG menu. Then scroll down to and select the MMULT function, which will enable us to premultiply our solutions vector **s** by matrix $\mathbf{C}^{-1}$. The dialogue box will prompt for two arrays. The first will be matrix $\mathbf{C}^{-1}$ in cells A4:B5. Then hit the Tab key and enter the cells for the second array C4:C5. Then hit the Ctrl, Shift, and Enter keys simultaneously to fill cells D4 and D5. The result will be vector **x** with elements −0.5 and 3.5. Thus, $x_1 = -0.5$ and $x_2 = 3.5$. The final spreadsheet will appear as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4	0.15	−0.1	10	−0.5	
5	−0.05	0.2	20	3.5	

The process of expanding this solution procedure to larger matrices is quite simple. First, be certain that each equation in the system is linear (no exponents other than 0 or 1 on the variables) and that the coefficient matrix is square. In many cases, the systems cannot be solved. Among these are the following: the coefficients matrix is not square; matrices do not conform for multiplication; or the coefficients matrix is singular. Consider the following fourth order system:

$$\begin{bmatrix} 8 & 4 & 2 & 10 \\ 2 & 4 & 1 & 12 \\ 0 & 4 & 2 & 16 \\ 5 & 6 & 8 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix},$$

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{s}.$$

Now, examine the following spreadsheet, which is used to solve the system:

	A	B	C	D	E	F	G
1	8	4	2	10			
2	2	4	1	12			
3	0	4	2	16			
4	5	6	8	20			
5							
6	0.235294	−0.29412	0.147059	−0.05882	10	−1.470588	
7	−0.52941	1.578431	−1.12255	0.215686	20	1.2254902	
8	−0.11765	−0.01961	−0.15686	0.196078	30	1.5686275	
9	0.147059	−0.39216	0.362745	−0.07843	40	1.372549	

Thus, $x_1 = -1.470588$, $x_2 = 1.2254902$, $x_3 = 1.5686275$, and $x_4 = 1.372549$.