CHAPTER NINE
INTEGRAL CALCULUS

9.1 ANTIDIFFERENTIATION AND THE INDEFINITE INTEGRAL (Background reading: section 8.3)

The derivative and the integral are the two most essential concepts from calculus. One might interpret the derivative, $f'(x)$, of a function $f(x)$ to be the slope of the curve plotted by that function. An analogous interpretation of the integral $\int f(x)\mathrm{d}x$ is the area $F(x)$ under a curve plotting the function $f(x)$. Thus, integrals are most useful for finding areas under curves. Similarly, they are useful for determining expected values and variances based on continuous probability distributions. As the Σ operator is used for summing countable numbers of objects, integrals are used for performing summations of uncountably infinite objects.

Integral calculus is also useful for analyzing the behavior of variables (such as cash flows) over time. A function $f(x)$ known as a differential equation might describe the rate of change of variable $f(x)$ over time; the solution to this differential equation, $F(x)$, describes the path itself over time. For example, $f(x)$ might describe the change in value or profit of an investment over time, while $F(x)$ provides its actual value.

Integrals of many functions can be determined by using the process of *antidifferentiation*, which is the inverse process of differentiation. If $F(x)$ is a function of x whose derivative equals $f(x)$, then $F(x)$ is said to be the antiderivative, or integral, of $f(x)$, written as follows:

$$F(x) = \int f(x)\mathrm{d}x. \tag{9.1}$$

The integral sign, $\int$, is used to denote the antiderivative of the *integrand* $f(x)$; the *indefinite integral* is denoted by $\int f(x)\mathrm{d}x$. The following is implied by equation (9.1):

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x). \tag{9.2}$$

Consider the following function: $f(x) = 4x^3$. The function for which $f(x)$ is the derivative is $F(x)$, the antiderivative of $f(x)$. The antiderivative $F(x)$ of $f(x) = x^4 + k$ is $4x^3$. Therefore, $f(x) = 4x^3$ is the derivative of $F(x) = x^4 + k$, where k is simply any real-valued constant:

$$\frac{dF(x)}{dx} = \frac{d(x^4 + k)}{dx} = 4x^3.$$

Thus, the derivative of the function $F(x)$ is the original function $f(x)$, implying that $F(x)$ is the antiderivative of $f(x)$. The constant of integration k must be included in the antiderivative. Thus, all of the following could be antiderivatives of $4x^3$: $F(x) = x^4 + 77$, $F(x) = x^4 + 6$, and $F(x) = x^4 + 1.25$. It is important for the antiderivative computation to be able to accommodate any of these possible constant values k.

The following are a few of the rules that apply to the computation of indefinite integrals (where k is a real-valued constant).

Let $f(x) = x^n$:

$$\int f(x)dx = \frac{x^{n+1}}{n+1} + k \qquad \text{for } n \neq -1. \tag{9.3}$$

Equation (9.3) is the polynomial (power) rule for finding antiderivatives.

$$\int Kf(x)dx = K\int f(x)dx, \tag{9.4}$$

where K is a constant.

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx; \tag{9.5}$$

$$\int \frac{1}{x}dx = \ln x + k. \tag{9.6}$$

The rule given by equation (9.6) is useful for many growth models. The following rule is particularly important for time value and valuation models:

$$\int e^{nx}dx = \frac{1}{n}e^{nx} + k \qquad \text{for } n \neq 0. \tag{9.7}$$

Other rules are provided in appendix 9.A.

9.2 Riemann Sums
(Background reading: section 9.1)

Consider a function $y = f(x)$. Suppose that we wish to find the area under a curve represented by this function over the range from $x = a$ to $x = b$. The lower limit of integration is said to be a; the upper limit of integration is said to be b. We will first show

how to find the area under a curve by demonstrating a method similar to one suggested by the Greek mathematician Archimedes in the third century B.C.E. This method was formalized by Bernhard Riemann in the mid-1800s and is now particularly useful for computer-based evaluations of integrals. The Riemann sum is also most useful for evaluating integrals of functions for which antiderivatives do not exist.

Consider the function $f(x) = 10x(1 - 0.1x)$. Suppose that we wish to find the area under the curve represented by this function over the range from $x = 0$ to $x = 1$. The method of Riemann sums divides the area under the curve into a number of rectangles, as in figure 9.1. Data for figure 9.1 are given in table 9.1. This curve has been divided into ten segments of width $x_i - x_{i-1} = 1/10$. The height of each rectangle is $y_i = f(x_i)$.

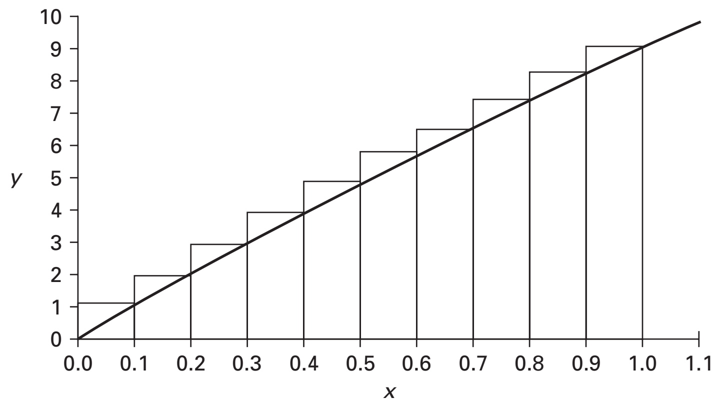

Figure 9.1 Finding the area under a curve using Riemann sums: $y = 10x(1 - 0.1x)$. When $x_i - x_{i-1} = 0.1$, the sum of the areas of the ten rectangles equals 5.115. As the number of rectangles approaches infinity, and their widths approach zero, the sum of their areas will approach $4\frac{2}{3}$.

Table 9.1 The area under the curve represented by $y = 10x(1 - 0.1x^2)$

Data point i	y_i	x_i	$x_i - x_{i-1}$	$y_i \cdot (x_i - x_{i-1})$
1	0.99	0.1	0.1	0.099
2	1.96	0.2	0.1	0.196
3	2.91	0.3	0.1	0.291
4	3.84	0.4	0.1	0.384
5	4.75	0.5	0.1	0.475
6	5.64	0.6	0.1	0.564
7	6.51	0.6	0.1	0.651
8	7.36	0.8	0.1	0.736
9	8.19	0.9	0.1	0.819
10	9.00	1.0	0.1	0.900
				$\Sigma[y_i \cdot (x_i - x_{i-1})] = 5.115$

Thus, the area of each rectangle is $f(x_i)(x_i - x_{i-1})$. The Riemann approximation for the area in the range from $x = 0$ to $x = 1$ based on ten rectangles is

$$\int_0^1 10x(1-0.1x)\mathrm{d}x \approx \sum_{i=1}^{10} f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{10} 0.1 \cdot 10x_i(1-0.1x_i) = 5.115.$$

Notice that the areas of rectangles in figure 9.1 do not correspond exactly with the areas of the curve that they are intended to simulate. Finer estimates of the area under the curve may be obtained by increasing the numbers of rectangles while decreasing their widths. We will continue to do this until the number of rectangles is sufficiently large to produce the desired level of accuracy. Generally, more rectangles of narrower widths lead to more accurate integral estimation.

The method of Riemann sums requires that we find the area of each of n rectangles. Each of these rectangles, which are arranged sequentially, will have a width of $x_i - x_{i-1}$. The rectangle width $x_i - x_{i-1} = 1/n$ will approach zero as the number of rectangles approaches infinity. The rectangle height will be $f(x^*)$, where x^* is some value between x_i and x_{i-1} (here we assume $x^* = x_i$).

To obtain increasingly finer area estimates, the number of these rectangles under the curve will approach infinity, and the width of each of these rectangles will approach (though not quite equal) zero. The area of each of these rectangles (where the product is nonnegative) is simply the product of its height and width:

$$\lim_{(x_i - x_{i-1} \to 1/n)} f(x^*) \cdot (x_i - x_{i-1}). \tag{9.8}$$

Thus, the area of a region extending from $x = a$ to $x = b$ under a curve can be found with the use of the definite integral over the interval from $x = a$ to $x = b$ as follows:

$$\int_a^b f(x)\mathrm{d}x = \lim_{\substack{n\to\infty \\ x_i - x_{i-1} \to 1/n}} \sum_{i=1}^{n} f(x^*)(x_i - x_{i-1}). \tag{9.9}$$

The right-hand side of equation (9.9) is the *Riemann sum*. The width of each rectangle equals $x_i - x_{i-1} = (b - a)/n \to 0$ and the height of each rectangle equals $f(x^*)$. We can use the Riemann sum to find the area under the curve in the example presented above as follows:

$$\int_0^1 10x(1-0.1x)\mathrm{d}x = \lim_{\substack{n\to\infty \\ x_i - x_{i-1} \to 1/n}} \sum_{i=1}^{n} 10x^*(1-0.1x^*)(x_i - x_{i-1}). \tag{A}$$

Since $b - a$ equals 1, each $x_i - x_{i-1}$ will equal $1/n$, and we obtain

$$\int_0^1 10x(1-0.1x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{10x_i(1-0.1x_i)}{n}. \tag{B}$$

Next, we note that our initial x_i value equals zero and that each x_i value equals i/n, since our units of increase are $1/n$. The counter i represents the number of increases accounted for at some point i in the summation. This enables us to obtain

$$\int_0^1 10x(1-0.1x)\mathrm{d}x = 0 + 10\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i}{n}\cdot\left(1-0.1\frac{i}{n}\right)}{n}$$

$$= 10\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2}\left(1-0.1\frac{i}{n}\right) = 10\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n^2}-0.1\frac{i^2}{n^3}\right). \quad \text{(C)}$$

The results of the series in equation (C) are well known and may be verified by induction:

$$\sum_{i=1}^{n}\frac{i}{n^2} = \frac{n(n+1)/2}{n^2}, \qquad \sum_{i=1}^{n}\frac{i^2}{n^3} = \frac{n(n+1)(2n+1)/6}{n^3}.$$

The results of these series are used to simplify equation (C):

$$\int_0^1 10x(1-0.1x)\mathrm{d}x = 10\cdot\left(\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2} - 0.1\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{n^3}\right)$$

$$= 10\cdot\left(\lim_{n\to\infty}\frac{n(n+1)/2}{n^2} - 0.1\cdot\lim_{n\to\infty}\frac{n(n+1)(2n+1)/6}{n^3}\right)$$

$$= 5\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right) - \frac{10\cdot 0.1}{6}\cdot\lim_{n\to\infty}\left(2+\frac{1}{n^2}+\frac{3}{n}\right) = 5-\tfrac{2}{6} = 4\tfrac{2}{3}. \quad \text{(D)}$$

Thus, as n approaches ∞, it is easy to see that the area under the curve extending from $x = 0$ to $x = 1$ approaches $4\frac{2}{3}$.

The use of Riemann sums to determine the precise area under a curve within a defined region involves the summation of areas of an infinite number of rectangles of infinitesimal width which lie within this area. We have calculated the area under our curve here and were able to do so precisely because we were able to easily simplify two infinite series. However, this process can be quite time-consuming, or must serve as only a finite approximation, when the series cannot be simplified. On the other hand, sums to reasonably large finite numbers can often provide an acceptable level of accuracy and can be a most useful means to obtain numerical values for integrals. This is particularly true when the integral to be evaluated has no antiderivative. We will discuss spreadsheet evaluations of integrals in appendix 9.B. Reading through this appendix may be quite helpful in understading the application of Riemann sums.

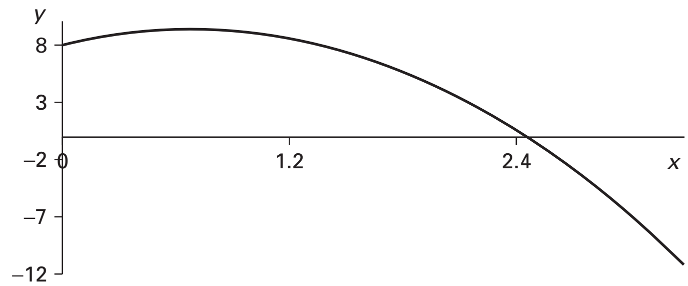

Figure 9.2 The area between the curve and the horizontal axis: $y = -3x^2 + 4x + 8$.

9.3 Definite Integrals and Areas (Background reading: sections 9.1 and 9.2)

Another, more elegant method for integration makes use of the Fundamental Theorem of Integral Calculus, based on a brilliant insight by Sir Isaac Barrow. This theorem is stated as follows:

If $f(x)$ is a continuous function within the range from $x = a$ to $x = b$, and $F(x)$ is the antiderivative of $f(x)$, the following must hold:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b \tag{9.10}$$

Thus, we may use the Fundamental Theorem of Integral Calculus to find the area under the function $f(x) = 10x(1 - 0.1x)$ by using antiderivatives as follows:

$$\begin{aligned}
&\int_0^1 10x(1 - 0.1x)\mathrm{d}x = \int_0^1 10x - x^2 \\
&= F(1) - F(0) = (5x^2 - \tfrac{1}{3}x^3)\Big|_0^1 \\
&= (5 \cdot 1^2 - \tfrac{1}{3} \cdot 1^3 + k) - (5 \cdot 0^2 - \tfrac{1}{3} \cdot 0^3 + k) \\
&= 5 - \tfrac{1}{3} = 4\tfrac{2}{3}.
\end{aligned} \tag{E}$$

Notice that the constants of integration k canceled out. Essentially, we found the antiderivative of our function at a (or 0), then subtracted this antiderivative from the antiderivative of our function at b (or 1).

Consider a second function, $y = -3x^2 + 4x + 8$, represented by figure 9.2. Suppose that we wished to find the area between this curve and the horizontal axis within the range from $x = 0$ to $x = 3$. Again, we may use the Fundamental Theorem of Integral Calculus to find the area under the curve by using antiderivatives as follows:

$$\int_0^3 (-3x^2 + 4x + 8)\mathrm{d}x = (-x^3 + 2x^2 + 8x)\Big|_0^3$$

$$= (-3^3 + 2 \cdot 3^2 + 8 \cdot 3 + k) - (-0^3 + 2 \cdot 0^2 + 8 \cdot 0 + k)$$

$$= -27 + 18 + 24 = 15. \quad \text{(A)}$$

The area between this curve and the horizontal axis net of the area under the curve but above the horizontal axis within the range $x = 0$ to $x = 3$ equals 15.

APPLICATION 9.1: CUMULATIVE DENSITIES (Background reading: sections 2.4 and 9.3)

A probability density function is a theoretical model for a frequency distribution. The (density at x^*) · dx might be regarded as the probability that a continuous random variable x lies between x^* and $x^* + \mathrm{d}x$ as $\mathrm{d}x \to 0$.[1] Thus, in a sense, the density function may be used to determine the probability $p(x_i)$ that a continuous random variable x_i will be exactly equal to a constant x^*. However, it is important to note that because the continuous random variable x_i may assume any one of an infinity of potential values, the probability that the function assumes any particular exact value x^* approaches zero ($p(x_i)\mathrm{d}x \to 0$). A continuous probability distribution $\mathrm{P}(x)$ may be used to determine the probability that a randomly distributed variable will fall within a given range or below a given value. Among the continuous probability distributions used by statisticians are the normal distribution, the uniform distribution, and the gamma distribution. A density function $p(x)$ can be computed based on a differentiable distribution function $\mathrm{P}(x)$ as follows:

$$p(x) = \frac{\mathrm{dP}(x)}{\mathrm{d}x} = \mathrm{P}'(x). \quad (9.11)$$

This implies that the distribution function $\mathrm{P}(x)$ may be found from the density function:

$$\mathrm{P}(x) = \int p(x)\mathrm{d}x \quad (9.12)$$

Consider a very simple density function $\{p(x) = 6(x - x^2)$ for $0 \le x \le 1$ and 0 elsewhere$\}$ for a particular randomly distributed variable x_i. From this density function, we can obtain a distribution function by integrating as follows:

$$\mathrm{P}(x) = \int p(x)\mathrm{d}x = \int 6[x - x^2]\mathrm{d}x = 6[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3]. \quad \text{(A)}$$

[1] The function $p(x)$ is a continuous version of P_i, which was used for the probability associated with a particular outcome i drawn from a discrete set of potential outcomes. See sections 5.5 and 5.6.

Notice that $P(0) = 0$ and $P(1) = 1$ and that $0 \le p(x)$ for all x, as one would expect for any density function ranging from zero to one. Now, suppose that the potential or random return r_i for a given stock is expected to range from 0 to 25%. Further suppose that potential returns track the random continuously distributed variable x_i whose probability distribution function is given by equation (A) ranging from 0 to 1. More specifically, the return on the stock is $r_i = f(x_i) = 0.25x_i$, or 25% of the value of this randomly distributed variable. The stock's return will always be 25% of the level of the random variable. We can use a definite integral to determine the probability that the random variable x_i is less than some constant x^*; this probability will be the same as for r_i being less than $0.25x^*$. The distribution function for the random variable is simply the cumulative density function. For example, we determine the probability that x_i will be less than 0.5 and that r_i will be less than 0.125 as follows:

$$P(x_i < 0.5) = P(r_i < 0.125) = \int_0^{0.5} p(x)\mathrm{d}x = \int_0^{0.5} 6[x - x^2]\mathrm{d}x$$
$$= 6[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3]\Big|_0^{0.5} = 6[\tfrac{1}{2}(0.5^2) - \tfrac{1}{3}(0.5^3)] - 6[\tfrac{1}{2}(0^2) - \tfrac{1}{3}(0^3)]$$
$$= 0.5 - 0 = 0.5. \tag{B}$$

Note that the lower limit of integration is zero because the density function is nonzero only over the interval from zero to one. Thus, there is a 50% probability that x_i will be less than 0.5 and that r_i will be less than 0.25.

We can also use definite integrals to determine the probability that the random variable will fall within a specified range. For example, we can integrate the density function $p(x)$ over the interval from 0.2 to 0.5 to determine the probability that x_i will fall between 0.2 and 0.5 (and that r_i will range between 0.05 and 0.125:

$$P(0.2 < x_i < 0.5) = P(0.05 < r_i < 0.125) = \int_{0.2}^{0.5} p(x)\mathrm{d}x = \int_{0.2}^{0.5} 6[x - x^2]\mathrm{d}x$$
$$= 6[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3]\Big|_{0.2}^{0.5} = 6[\tfrac{1}{2}(0.5^2) - \tfrac{1}{3}(0.5^3)] - 6[\tfrac{1}{2}(0.2^2) - \tfrac{1}{3}(0.2^3)]$$
$$= 0.5 - 0.104 = 0.396. \tag{C}$$

The probability that the return will range between 0.05 and 0.125 is also equal to 0.396. We can just as easily determine that the probability that x_i will fall between 0.4 and 0.6 (and the probability that r_i will fall between 0.04 and 0.06):

$$P(0.4 < x_i < 0.6) = P(0.10 < r_i < 0.15) = \int_{0.4}^{0.6} p(x)\mathrm{d}x = \int_{0.4}^{0.6} 6[x - x^2]\mathrm{d}x$$
$$= 6[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3]\Big|_{0.4}^{0.6} = 6[\tfrac{1}{2}(0.6)^2 - \tfrac{1}{3}(0.6^3)] - 6[\tfrac{1}{2}(0.4^2) - \tfrac{1}{3}(0.4^3)]$$
$$= 0.648 - 0.352 = 0.296. \tag{D}$$

The normal density function is the most useful density function in finance:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}},$$

where μ and σ are parameters representing the mean and standard deviation of random variable x. Unfortunately, no closed-form solution exists for the following integral (see appendix 9.A):

$$\mathrm{P}(x) = \mathrm{N}(x) = \int_{-\infty}^{x^*} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \mathrm{d}y.$$

Riemann sums and polynomials are often constructed to evaluate these integrals (see appendix 9.B).

APPLICATION 9.2: EXPECTED VALUE AND VARIANCE (Background reading: sections 5.6, 5.7, and 9.2 and application 9.1)

Expected values and variances are very useful in characterizing probability distributions, random variables, and functions based on random variables. For our example, we will continue to use the density function from application 9.1: $p(x) = 6(x - x^2)$ for $0 \le x \le 1$ and 0 elsewhere. We will evaluate integrals of this density function to generate an expected value and a variance for our randomly distributed variable and the stock whose return tracks that random variable. To find the expected return, use the density function to weight each random return r_i:

$$\mathrm{E}[r_i] = \int_0^1 f(x_i)p(x_i)\mathrm{d}x = \int_0^1 0.25(x_i)p(x_i)$$

$$= \int_0^1 0.25(x_i)\cdot 6(x_i - x_i^2) = \int_0^1 1.5(x_i^2 - x_i^3) \qquad \text{(A)}$$

for $0 \le x \le 1$ and 0 elsewhere. Notice the similarity between $\mathrm{E}[r_i] = \int f(x_i)p(x_i)\mathrm{d}x$ for continuous functions and $\mathrm{E}[r_i] = \sum r_i p(i)$, that we used earlier for discrete probability functions. Assuming that the constant of integration equals zero, the indefinite integral of this density function for r is determined as follows:

$$\int f(x)p(x)\mathrm{d}x = 1.5[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4]\mathrm{d}x. \qquad \text{(B)}$$

The expected value of this random variable r is determined as follows:

$$
\begin{aligned}
\mathrm{E}[r] &= \int_0^1 f(x)p(x)\mathrm{d}x = 1.5[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4]\Big|_0^1 \\
&= 1.5[\tfrac{1}{3}\cdot 1^3 - \tfrac{1}{4}\cdot 1^4] - 1.5[\tfrac{1}{3}\cdot 0^3 - \tfrac{1}{4}\cdot 0^4] \\
&= 1.5[\tfrac{1}{3} - \tfrac{1}{4}] - 0 = 0.125. \qquad \text{(C)}
\end{aligned}
$$

Thus, the expected return for this security equals 0.125.

The variance of returns for a stock may be determined using the following:

$$
\begin{aligned}
\sigma^2 &= \mathrm{E}[r_i - \mathrm{E}[r]]^2 = \mathrm{E}[r_i^2] - \mathrm{E}[r]^2 \\
&= \left[\int_0^1 [f(x)]^2 p(x)\mathrm{d}x\right] - \left[\int_0^1 f(x)p(x)\mathrm{d}x\right]^2. \qquad \text{(D)}
\end{aligned}
$$

Again, notice the similarity of the top line of equation (D) to discrete variance formulas. The variance of returns for the stock in our example is determined as follows:

$$
\begin{aligned}
\sigma^2 &= \int_0^1 0.25^2x^2\cdot 6[x - x^2]\mathrm{d}x - 0.125^2 = \int_0^1 0.4[x^3 - x^4]\mathrm{d}x - 0.125^2 \\
&= 0.4[\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5]\Big|_0^1 - 0.125^2 = 0.4[\tfrac{1}{4} - \tfrac{1}{5}] - 0 - 0.125^2 \\
&= 0.02 - 0.015625 = 0.004375; \qquad \sigma = 0.066144. \qquad \text{(E)}
\end{aligned}
$$

Thus, the variance of returns in this distribution is 0.004375 and the standard deviation of returns is 0.066144.

APPLICATION 9.3: VALUING CONTINUOUS DIVIDEND PAYMENTS (Background reading: sections 4.5, 4.8, 9.2, and 9.3)

Many companies make dividend payments on a quarterly basis. However, payment calendars vary from firm to firm. An index simulating a portfolio of a large number of stocks paying dividends will probably reflect dividend payments scattered throughout the year. For example, a portfolio of securities intended to match the S&P 500 Index would probably receive dividends from companies on practically a daily or continuous basis (500 stocks, each receiving up to four dividend payments per 365-day year). It may be practical to model the dividend payment structure into such a portfolio as though the dividends were received continuously.

Suppose that we wished to value the dividend stream received by a fund over a five-year period. The fund contains a large number of stocks paying dividends staggered throughout the year. We will assume that this fund receives dividends on a continuous basis at a rate of \$100,000 per year, starting at time $t = 0$. The same dividend amount is received by the fund each day; the dividend stream is continuous. Further assume that these dividends will be discounted at a continuously compounded rate of $k = 5\%$

per year. The present value of all dividends received in any infinitesimal interval $[t,t + \mathrm{d}t]$ is as follows:

$$PV[t,t + \mathrm{d}t] = PV[0,t + \mathrm{d}t] - PV[0,t], \tag{9.12}$$

where $PV[t,t + \mathrm{d}t]$ equals the present value of dividends received during the interval $[t,t + \mathrm{d}t]$. Equation (9.12) implies that dividends received increase the value of the fund. Equation (9.12) describes the rate of change in the fund due to dividends received. The amount of dividend payment to be received at any infinitesimal time interval $\mathrm{d}t$ equals $f(t)\mathrm{d}t = 100{,}000\mathrm{d}t$. The present value of this sum equals $f(t)\mathrm{e}^{-kt}\mathrm{d}t = 100{,}000\mathrm{e}^{-0.05t}\mathrm{d}t$. Thus, the present value of dividends received over the infinitesimal interval $\mathrm{d}t$ is

$$PV[t,t + \mathrm{d}t] = PV[0,t + \mathrm{d}t] - PV[0,t] = f(t)\mathrm{e}^{-kt}\mathrm{d}t = 100{,}000\mathrm{e}^{-0.05t}\mathrm{d}t. \tag{A}$$

To find the present value of a sum received over a finite interval beginning with $t = 0$, one may apply the definite integral as follows:

$$PV[0,T] = \int_0^T f(t)\mathrm{e}^{-kt}\mathrm{d}t. \tag{9.13}$$

While equations (9.12) and (A) represent the rate of change or direction of the dividend stream value, equations (9.13) and (B) following represent the path or cumulative value. In our numerical example, one may find the present value of dividends to be paid to the fund from time 0 for five years as follows:

$$\begin{aligned} PV[0,5] &= \int_0^5 100{,}000\mathrm{e}^{-0.05t}\mathrm{d}t = 100{,}000\left[-\frac{\mathrm{e}^{-0.05t}}{0.05}\right]_0^5 \\ &= \frac{100{,}000}{0.05}(1 - \mathrm{e}^{-0.25}) = \$442{,}398. \end{aligned} \tag{B}$$

A useful application of this methodology is to the valuation of index contracts that are not dividend protected. Many indices reflect component stock prices but not dividends paid by component stocks. Since stock prices decline to reflect the value of dividends paid, the funds that include the stocks will decline in value to reflect the value of dividends paid. Investors taking positions in options and futures contracts in such funds will find that the fund values will decline as dividends are paid by component securities and must structure valuation models to account for dividends. For example, suppose that one can take a position on a contract to purchase a fund at time T. Any dividends to be received by the fund will be paid immediately to current fund investors as they are received.

Suppose that the fund described above is currently worth $2,000,000 to current investors and pays to its investors any dividends as they are received. Assume that a call option contract enabling the investor to purchase shares of the fund in five years will not enable the investor to obtain any dividends paid by the fund over the next five years. The portion of the fund's current value attributable to dividends to be paid over the next five years should be deducted from its overall value, since the investor exercising the option will not have a right to receive these dividends. The investor with the option contract to purchase the fund in five years may value the fund after dividends at $2,000,000 – $442,398 = $1,557,602 based on the present value of dividends that he will not be able to receive should he exercise his call option.

APPLICATION 9.4: EXPECTED OPTION VALUES (Background reading: section 9.3 and application 7.8)

The expected future value of a European call is equal to its expected value if it is exercized, $E[(S_n - X)|S_n - X]$, multiplied by the probability that it will be exercized, $P[S_n > X]$. If the range of potential stock prices is continuous, this might be written as follows:

$$E[c_n] = \int_X^{\infty} (S_n - X)p(S_n)dS_n, \tag{9.14}$$

where $p(S_n)$ is the density function for the stock's price. Note that the call's value is zero when $S_n < X$. The probability that the call will be exercised equals

$$P[S_n > X] = \int_X^{\infty} p(S_n)dS_n. \tag{9.15}$$

The expected value of the call conditioned on it being exercised is

$$E[c_n|S_n > X] = \frac{\int_X^{\infty} (S_n - X)p(x)dS_n}{\int_X^{\infty} p(S_n)dS_n}. \tag{9.16}$$

The expected value of the call (9.14) is simply the product of equations (9.15) and (9.16).

9.4 DIFFERENTIAL EQUATIONS

Financial economists and practitioners are often concerned with the development or change of a variable or asset over time. A *differential equation* might be structured to model the change (evolution or direction) of an asset over time. From this equation, a second equation (*solution*) might be derived to describe the asset's value (state or path) at a given point in time. A differential equation is defined as a function for which one or more derivatives exist. The solution of a differential equation is a function that, when substituted for the dependent variable in the differential equation, leads to an identity. The following is a simple differential equation and its solution involving dependent variable x and independent variable t:

$$\frac{dx}{dt} = t, \tag{A}$$

$$x = \tfrac{1}{2}t^2 + k. \tag{B}$$

We verify the solution to differential equation (A) by noting that it represents the derivative of x with respect to t in its solution (B). Equation (A) represents the change in variable x over time ($dx = tdt$). Note that this rate of change increases as t increases. Equation (B) represents the state or value of x at a given point in time t.

A differential equation is said to be *separable* if it can be rewritten in the form $g(x)dx = f(t)dt$. A separable differential equation written in this form can be solved by the following:

$$\int g(x)dx = \int f(t)dt. \tag{9.17}$$

The following is an example of a separable differential equation:

$$\frac{dx}{dt} = 0.05x.$$

In this equation, $f(t) = 0.05$ and $g(x) = 1/x$. To solve this equation, we first separate the variables as follows:

$$\frac{1}{x} \quad dx = 0.05 \quad dt,$$

$$g(x) \quad dx = f(t) \quad dt.$$

Next, we integrate both sides and to obtain a *general solution* for x:

$$\int \frac{dx}{x} = \int 0.05dt,$$

$$\int \frac{1}{x}dx = \int 0.05dt,$$

$$\ln x + k_1 = 0.05t + k_2,$$

$$\ln x = 0.05t - k_1 + k_2,$$

$$e^{\ln x} = e^{0.05t} \cdot e^{-k_1+k_2},$$

$$x = Ke^{0.05t} \qquad \text{where } e^{-k_1+k_2} = K.$$

The constant K can assume any value. Thus, the general solution for our differential equation could assume any of an infinity of values. A *particular solution* results when K assumes a specific value. In this case, one particular solution for x could be $x = x_0 e^{0.05t}$, where x_0 is the value of x when $t = 0$. This type of differential equation typical of those used for modeling growth.

APPLICATION 9.5: SECURITY RETURNS IN CONTINUOUS TIME (Background reading: section 9.4 and application 9.3)

Many valuation models are based on continuous time and continuous space. This means that the securities that they value evolve continuously over time (their prices can be observed at every instant) and their prices can take on any real number value. Suppose that the evolution of a stock's price could be described by the following separable differential equation:

$$\frac{dS_t}{dt} = \mu S_t. \tag{9.18}$$

The term μ represents the security's drift or mean instantaneous rate of return. Thus, the security's price change per infinitesimal unit of time equals μdt. This differential equation can be separated as follows:

$$\frac{dS_t}{S_t} = \mu dt. \tag{9.19}$$

The solution to this differential equation gives the security's price at a moment in time t. The solution to differential equation (9.18) can be obtained with the following integrals:

$$\int \frac{dS_t}{S_t} = \int \mu dt.$$

These integrals can be solved as follows:

$$\ln(S_t) + k_1 = \mu t + k_2.$$

We let $K = k_2 - k_1$ and write the anti-logs of the results of both sides, as

$$e^{\ln(S_t)} = e^{\mu t + K},$$

$$S_t = e^{\mu t} \cdot e^K. \tag{9.20}$$

Equation (9.20) represents a general solution to our differential equation (9.18). If we set e^K equal to the stock's price S_0 at time zero, the particular solution to equation (9.18) would be:

$$S_t = S_0 e^{\mu t}. \tag{9.21}$$

Differential equations such as (9.18) are very useful in the modeling of security prices and are adaptable to the modeling of stochastic (random) return processes.

Now, consider a security with value S_t in time t, generating returns on a continuous basis such that the security were to double in value every seven years. Suppose that the value of this security after ten years were \$100. What would have been the initial value S_0 of this security? We will use equation (9.18) as the security's return generating process:

$$\frac{dS_t}{S_t} = \mu dt.$$

The solution to this equation is given by equation (9.20):

$$S_7 = S_0 e^{7\mu} = 2S_0.$$

Thus, $\mu = \ln(2) \div 7 = 0.09902$. With this result, we can easily solve for the security's initial value:

$$S_0 = S_{10} e^{-10 \cdot 0.09902} = \$100 e^{-10 \cdot 0.09902} = \$371.50.$$

APPLICATION 9.6: ANNUITIES AND GROWING ANNUITIES (Background reading: section 9.4 and application 9.5)

Consider an investor who continuously collects dividends from his brokerage account at a rate of \$100,000 per year. The dividends are colleted in equal installments during each interval of time (day or smaller time period dt) during the year such that the installments can be modeled as being continuous. If dividend payments are to be discounted at an annual rate of 5%, what would be the present value of the dividend payment stream over a one-year period?

The amount of dividend payment to be received by the investor during any infinitesimal interval dt equals $f(t)dt = \$100{,}000dt$. The present value of this sum received at time t equals $f(t)e^{-kt}dt = \$100{,}000e^{-0.05t}dt$. To find the present value of the entire sum received over a finite interval beginning with $t = 0$, solve the following definite integral:

$$PV[0,T] = \int_0^T f(t)e^{-kt}dt,$$

$$PV[0,1] = \int_0^1 \$100{,}000e^{-0.05t}dt = \$100{,}000\left[-\frac{e^{-0.05t}}{0.05}\right]\Bigg|_0^1$$

$$= \frac{\$100{,}000}{0.05}(1 - e^{-0.05}) = \$97{,}541.15.$$

Now, consider a growing series of cash flows. We will derive a growing annuity formula with continuous cash flow streams. First, we structure and combine the growth and discount functions to be integrated over time:

$$PV[t,t + \mathrm{d}t] = CF_0 \mathrm{e}^{gt}\mathrm{e}^{-kt}\mathrm{d}t = CF_0 \mathrm{e}^{(g-k)t}\mathrm{d}t,$$

$$PV[0,T] = \int_0^n CF_t \mathrm{e}^{(g-k)t}\mathrm{d}t.$$

Integrating and canceling constants of integration, we obtain the following annuity function:

$$PV[0,n] = \int_0^n CF_0 \mathrm{e}^{(g-k)t}\mathrm{d}t = CF_0 \left[-\frac{\mathrm{e}^{(g-k)t}}{k-g} \right]\Bigg|_0^n$$

$$= \frac{CF_0}{k-g}(1 - \mathrm{e}^{(g-k)n}). \tag{9.22}$$

Note the similarity between the continuous growing annuity formula above and the discrete cash flow version below:

$$PV[0,n] = \sum_{t=0}^{n} CF_0 \frac{(1+g)^t}{(1+k)^t} = \frac{CF_0}{k-g}\left[1 - \frac{(1+g)^n}{(1+k)^n}\right].$$

Now, suppose that the dividend stream in the example presented above were growing at a continuously compounded rate of 2% per year. Assume that its initial annual rate were \$100,000. The present value of this stream over its first year is evaluated as follows:

$$PV[0,1] = \frac{100{,}000}{0.05 - 0.02}(1 - \mathrm{e}^{(0.02-0.05)\cdot 1}) = 98{,}514.89.$$

EXERCISES

9.1. Integrate each of the following functions over x:

(a) $f(x) = 0$;
(b) $f(x) = 5$;
(c) $f(x) = 15x^2$;
(d) $f(x) = 15x^2 + 5$;
(e) $f(x) = \mathrm{e}^x$;
(f) $f(x) = 0.5\mathrm{e}^{0.5x}$;

(g) $f(x) = 5^x \ln(5)$;
(h) $f(x) = 1/x$.

9.2. Solve each of the following definite integrals:

(a) $$\int_0^1 x\,dx;$$

(b) $$\int_2^4 (x + 5)\,dx;$$

(c) $$\int_0^{20} 100{,}000e^{0.10t}\,dt.$$

9.3. Calculate Riemann sums for each of the three functions in problem 9.2. Let $n = 5$ (five boxes) for each part.

9.4. Assume that the density function $\{p(x_i) = P'(x_i)\}$ for a randomly distributed variable x is given by the following: $p(x) = 1.5x^2 + x$ for $0 \le x \le 1$ and 0 elsewhere.

(a) Find the distribution function $P(x)$ based on $p(x)$.
(b) Find the expected value of x in the range 0.2 to 1.
(c) Find the expected value of x in the range 0 to 0.2.
(d) Find the expected value of x in the range 0 to 1.
(e) Find the variance of x in the range 0 to 1.

9.5. A stock's terminal value S has a uniform distribution; that is, it is equally likely to assume any value in the range (0–100) and will not assume any value outside of this range. The random variable x on which this stock's value is based has a density function $p(x) = 1$ for $0 \le x \le 1$ and 0 elsewhere. The stock's random terminal value is $f(x) = 100x$.

(a) Find the distribution function $P(x)$ for $p(x)$.
(b) Find the expected value of the stock's terminal S value assuming that it will fall within the range 50–100.
(c) Find the expected value of the stock's terminal S value assuming that it will fall within the range 0–50.
(d) Find the expected value of S in the range 0–100.
(e) Find the variance of S in the range 0–100.
(f) What would be the expected future cash flow (contingent on its exercise) of a call option written on this stock if its exercise price were \$50? That is, what is the expected cash flow of the option conditional on its exercise?

(g) What would be the expected cash flow of a call option written on this stock if its exercise price were \$50? That is, what is the unconditional expected cash flow associated with this option?

(h) If the appropriate continuously compounded discount rate for all assets equals 0.10 and the cash flows described in parts (a) through (g) were to be paid in one year, what would be the present value of the call option?

9.6. Assume that the density function p_f for a randomly distributed variable is given by $p_f(x) = 4x^3$ for $0 \leq x \leq 1$ and 0 elsewhere. A second density function p_g for a randomly distributed variable is given by $p_g(x) = (3x^4 + 0.8x)$ for $0 \leq x \leq 1$ and 0 elsewhere. Find $\mathrm{P}_f(x)$ and $\mathrm{P}_g(x)$.

9.7. A pension fund collects \$3,000,000 in dividends per year from its various securities. The dividends are received in equal installments during each interval (day or dt) during the year such that they can be modeled as being continuous. If dividends are to be discounted at an annual rate of 6%, what would be the present value of the dividend stream over the next ten years?

9.8. A dividend stream with an initial annual rate of \$10,000 will grow at a continuously compounded rate of 3% per year. Cash flows are discounted at 5%. Find the present value of this stream over its first two years.

9.9. An individual wishes to retire with \$1,000,000 invested at an annual interest rate of 6%. She will withdraw \$75,000 per year for living expenses. Assume that she will make withdrawals continuously throughout the year.

(a) Let PV designate the present value of the account, let FV_t designate the value of the account at time t, let PMT designate the payment made from the account during each year, and let i designate the interest rate. Devise an appropriate differential equation describing the rate of change in the retiree's account during any given time period.

(b) Solve the differential equation to find the balance of the account at any time T.

(c) Based on the solution to the relevant differential equation and the numbers given in the example, how much money would the retiree have in his account after ten years?

(d) If the retiree continues to spend \$100,000 per year, at what point (how many years) will she run out of money?

9.10. Work through each of the parts of exercise 9.9 assuming that the investor wished for her payments to start at an annual rate of \$75,000 but increase at an annual rate of 2% to cover anticipated inflation.

APPENDIX 9.A RULES FOR FINDING INTEGRALS

Let $f(x) = x^n$:

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + k \qquad \text{for } n \neq -1;$$

$$\int Kf(x)\mathrm{d}x = K\int f(x)\mathrm{d}x;$$

$$\int [f(x) + g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x;$$

$$\int \frac{1}{x}\mathrm{d}x = \ln|x| + k;$$

$$\int \mathrm{e}^{nx}\mathrm{d}x = \frac{1}{n}\mathrm{e}^{nx} + k \qquad \text{for } n \neq 0;$$

$$\int a^x \mathrm{d}x = \frac{1}{\ln a}a^x + k \qquad \text{for } a > 0, a \neq 1;$$

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(x)\mathrm{d}x;$$

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x \text{ (integration by parts);}$$

$$\int f(x)\mathrm{d}x = \int f[g(t)]g'(t)\mathrm{d}t \text{ (change of variable of integration).}$$

The following integrals have no closed-form solutions and are often estimated using Riemann sums or polynomial approximations:

$$\int \mathrm{e}^{x^2}\mathrm{d}x,$$

$$\int \mathrm{e}^{-x^2}\mathrm{d}x,$$

$$\int \frac{\mathrm{e}^x}{x}\mathrm{d}x,$$

$$\int \frac{1}{\ln x}\mathrm{d}x.$$

APPENDIX 9.B RIEMANN SUMS ON A SPREADSHEET

(Background reading: section 9.2)

The Riemann sum was introduced in section 9.2 as a means of integrating a function without the need to find an antiderivative. This tool can be particularly useful for spreadsheet files that require the computation of an integral. An approximation using a Riemann sum of a finite number of rectangles with finite widths may be obtained fairly efficiently on a spreadsheet. As n gets larger, computational accuracy will tend to increase, although computational time will increase.

Suppose that we wish to evaluate the function $F(x) = \int_0^1 3(x - x^2)\mathrm{d}x$ using a Riemann approximation. The lower limit of integration is 0; the upper limit of integration is 1. We will start by dividing the area under the curve into 20 rectangles. Data for these rectangles are given in table 9.B.1. Each of these rectangles, which are numbered sequentially, will have a width of $x_i - x_{i-1} = 1/20 = 0.05$ and a height of $f(x^*) = 3(x - x^2)$. We see that the area of each of these rectangles is simply the product of its height and width:

$$\text{area} = f(x_i^*) \cdot (x_i - x_{i-1}) = f(x_i^*) \cdot 0.05. \tag{9.B.1}$$

Thus, the area of a region extending from $x = a$ to $x = b$ under this curve can be approximated with the following Riemann sum of 20 rectangle areas:

$$\text{area} = \sum_{i=1}^{\mathrm{n}} f(x_i^*) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{20} 3 \cdot (0.05i - (0.05i)^2) \cdot 0.05. \tag{9.B.2}$$

For example, the first rectangle (as all the others) has a width equal to 0.05. Its height is $3(0.05 - 0.05^2) = 0.1425$. Hence its area is $0.05 \cdot 0.1425 = 0.007125$. The second box has a height equal to $3(0.1 - 0.01) = 0.27$, giving it an area of 0.0135. We continue this process until $i = 20$ and sum the rectangle areas. This provides us with a total area approximation equal to 0.49875, slightly less than the exact area of 0.5 found by antidifferentiating. The accuracy of our estimate is improved by increasing n, which increases the number of rectangles while decreasing the width of each.

An important application of Riemann sums is to the evaluation of integrals for which no antiderivatives exist (see appendix 9.A). For example, consider the normal curve:

$$F(Y) = \mathrm{N}(Y) = \int_{-\infty}^{Y} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(y-\mu)^2}{2\sigma^2}} \mathrm{d}y.$$

No antiderivative exists for this function. However, we can use Riemann sums quite easily to approximate this integral, as in table 9.B.2. Suppose that we wished to find the area under the normal curve extending from 0 to 2. The lower limit of integration will be 0 and the upper limit will be 1. Let us assume that our normal curve has a mean of 0 and a standard deviation equal to 1. Again, we will divide the area between 0 and 2 under the normal curve into 20 rectangles. The width of each rectangle equals 0.1. The height of each equals:

Table 9.B.1 Riemann approximation for $F(x) = \int_0^1 3(x - x^2)dx$

	A	B	C	D	E	F	G
1	$F(x) = \int_0^1 3(x - x^2)dx$	Entries	X(i)	X(i)-X(i-1)	f(X(i))	f(x(i))(X(i)-X(i-1))	SUM[f(x(i))(X(i)-X(i-1))]
2	Enter Function Here (In Cell B2):	0	0	N/A	0	N/A	N/A
3	Enter Lower Limit of Integration:	0	0.05	0.05	0.1425	0.007125	0.007125
4	Enter Upper Limit of Integration:	1	0.1	0.05	0.27	0.0135	0.020625
5	Enter Number of Boxes Under Curve:	20	0.15	0.05	0.3825	0.019125	0.03975
6			0.2	0.05	0.48	0.024	0.06375
7	Upper Minus Lower Limit of Integration:	1	0.25	0.05	0.5625	0.028125	0.091875
8	Width of Each Box:	0.05	0.3	0.05	0.63	0.0315	0.123375
9			0.35	0.05	0.6825	0.034125	0.1575
10	Directions:		0.4	0.05	0.72	0.036	0.1935
11	1. Enter Function in Cell B2.		0.45	0.05	0.7425	0.037125	0.230625
12	2. Type the same function in Cell E2.		0.5	0.05	0.75	0.0375	0.268125
13	3. Copy down the function in E2.		0.55	0.05	0.7425	0.037125	0.30525
14	4. Enter values for B3, B4 and B5.		0.6	0.05	0.72	0.036	0.34125
15	5. Definite integral estimate corresponds to the cell in Column C holding the upper limit of integration. The integral is located in Column G in the cell corresponding to the upper limit of integration in column C. In this example, see Cell G22.		0.65	0.05	0.6825	0.034125	0.375375
16			0.7	0.05	0.63	0.0315	0.406875
17			0.75	0.05	0.5625	0.028125	0.435
18			0.8	0.05	0.48	0.024	0.459
19			0.85	0.05	0.3825	0.019125	0.478125
20			0.9	0.05	0.27	0.0135	0.491625
21			0.95	0.05	0.1425	0.007125	0.49875
22	Note: Cell A1 may be entered if desired, but it serves no computational role.		1	0.05	−7E-16	−3.33067E-17	0.49875
23							

Table 9.B.2 Riemann approximation for the area under the normal curve

	A	B	C	D	E	F	G
1	F(x) = (1/(sigma * 2 * pi)^.5 * e^-(((x-u)^2)/2)dx	Entries	X(i)	X(i)-X(i-1)	f(X(i))	f(x(i))(X(i)-X(i-1))	SUM[f(x(i))(X(i)-X(i-1))]
2	Enter Mean Here	0	0	N/A	0.3989	N/A	N/A
3	Enter Lower Limit of Integration:	0	0.1	0.1	0.397	0.039695335	0.039695335
4	Enter Upper Limit of Integration:	2	0.2	0.1	0.391	0.039104348	0.078799683
5	Enter Number of Boxes Under Curve:	20	0.3	0.1	0.3814	0.038138858	0.116938541
6	Enter Standard Deviation Here:	1	0.4	0.1	0.3683	0.036827088	0.153765629
7	Upper Minus Lower Limit of Integration:	2	0.5	0.1	0.3521	0.035206604	0.188972233
8	Width of Each Box:	0.1	0.6	0.1	0.3332	0.033322527	0.222294761
9			0.7	0.1	0.3123	0.031225456	0.253520217
10	Directions:		0.8	0.1	0.2897	0.028969214	0.282489431
11	1. Enter Mean in B2 and Standard Deviation		0.9	0.1	0.2661	0.026608579	0.309098009
12	in B6.		1	0.1	0.242	0.024197121	0.333295131
13	2. Enter the other values for Cells B3 to B5.		1.1	0.1	0.2179	0.021785262	0.355080392
14	3. Definite integral estimate corresponds		1.2	0.1	0.1942	0.019418645	0.374499037
15	to the cell in Column C holding the		1.3	0.1	0.1714	0.017136894	0.391635931
16	upper limit of integration. The integral		1.4	0.1	0.1497	0.014972777	0.406608708
17	is located in Column G in the cell		1.5	0.1	0.1295	0.012951786	0.419560494
18	corresponding to the upper limit of		1.6	0.1	0.1109	0.011092106	0.4306526
19	integration in column C. Here, the area is		1.7	0.1	0.094	0.009404927	0.440057526
20	Cell G22.		1.8	0.1	0.079	0.007895032	0.447952558
21	Riemann Sum for the Cumulative Normal		1.9	0.1	0.0656	0.006561595	0.454514153
22	Density Function		2	0.1	0.054	0.005399108	0.459913261
23							

$$f(x^*) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(f(x^*)-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{\frac{-(f(x^*))^2}{2}}.$$

We find areas just as in the previous example and sum them. Our area estimate is given in cell G22 as 0.464311629 in table 9.B.2. Hence, approximately 46% of the area under the normal curve lies between the mean and one standard deviation to the right of the mean. An alternative estimate of 0.97725 is obtained with the NORMDIST function in the Statistical submenu of the Paste Function $\boldsymbol{f_x}$. While both are estimates, the accuracy of the Riemann sum can be improved substantially by using n equal to 1,000 or a much larger value.