CHAPTER SEVEN

Elements of Matrix Mathematics

7.1 An Introduction to Matrices

Investors frequently encounter situations involving numerous potential outcomes, many discrete periods of time and large numbers of diverse securities. The process of analyzing such large quantities of inputs and functions can involve extraordinary numbers of repetitive and time-consuming computations. Even the preparation and organization of inputs and data can be enormously complex. Matrices and matrix mathematics are among the tools available to the analyst for the systematizing laborious operations and calculations.

A *matrix* is simply an ordered rectangular array of numbers. A matrix is an entity that enables one to represent a series of numbers as a single object, thereby providing for convenient systematic methods for completing large numbers of repetitive computations. Rules of matrix arithmetic and other matrix operations are often similar to rules of ordinary arithmetic and other operations, but they are not always identical. In this text, matrices will usually be denoted with bold upper-case letters. When the matrix has only one row or one column, bold lower-case letters will be used for identification. The following are examples of matrices:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 7 & 4 \\ 8 & -5 & 9 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & -3 \\ \frac{3}{4} & -\frac{1}{2} \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} \frac{3}{5} \end{bmatrix}.$$

The dimensions of a matrix are given by the ordered pair $m \times n$, where m is the number of rows and n is the number of columns in the matrix. The matrix is said to be of *order* $m \times n$ where, by convention, the number of rows is listed first. Thus, $\mathbf{A}$ is 3×3, $\mathbf{B}$ is 2×2, $\mathbf{c}$ is 3×1, and $\mathbf{d}$ is 1×1. Each number in a matrix is referred to as an element. The symbol $a_{i,j}$ denotes the element in row i and column j of matrix $\mathbf{A}$, $b_{i,j}$ denotes the element in row i and column j of matrix $\mathbf{B}$, and so on. Thus, $a_{3,2}$ is -5 and $c_{2,1} = 5$.

QRMC07 9/17/01 4:45 PM Page 116

There are specific terms denoting various types of matrices. Each of these particular types of matrices have useful applications and unique properties for working with. For example, a *vector* is a matrix with either only one row or one column. Thus, the dimensions of a vector are $1 \times n$ or $m \times 1$. Matrix **c** above is a column vector; it is of order 3×1. A $1 \times n$ matrix is a row vector with n elements. The column vector has one column and the row vector has one row. A *scalar* is a matrix with exactly one element. Matrix **d** is a scalar. A *square matrix* has the same number of rows and columns ($m = n$). Matrix **A** is square and of order 2. The set of elements extending from the upper-leftmost corner to the lower-rightmost corner in a square matrix are said to be in the *principal diagonal*. For each of these elements $i_{i,j}$, $i = j$. The principal diagonal elements of square matrix **A** are $a_{1,1} = 4$, $a_{2,2} = 7$, and $a_{3,3} = 9$. Matrices **B** and **d** are also square matrices.

A *symmetric matrix* is a square matrix where $c_{i,j}$ equals $c_{j,i}$ for all i and j; that is, the ith element in each row equals the jth element in each column. Scalar **d** and matrices **H**, **I**, and **J** below are all symmetric matrices. A *diagonal matrix* is a symmetric matrix whose elements off the principal diagonal are zero, where the *principal diagonal* contains the series of elements for which $i = j$. Scalar **d** and matrices **H** and **I** below are all diagonal matrices. An *identity* or *unit* matrix is a diagonal matrix consisting of ones along the principal diagonal. Both matrices **H** and **I** following are diagonal matrices; **I** is the 3×3 identity matrix:

$$\mathbf{H} = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 1 & 7 & 2 \\ 7 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}.$$

APPLICATION 7.1: PORTFOLIO MATHEMATICS
(Background reading: sections 2.9, 6.2, 6.3, and 7.1)

We saw in section 6.3 that computing returns and variances for portfolios with large numbers of securities often involves large numbers of repetitive calculations. The use of matrices provides a means of organizing, systemizing, and generally simplifying these series of calculations. We use standard rules of matrix operations to perform many useful computations. Consider a portfolio comprising three securities with the following characteristics and weights:

$$\begin{array}{llll} \mathrm{E}[R_1] = 0.12, & \sigma_1 = 0.30, & w_1 = 0.2, & \sigma_{1,2} = 0.01, \\ \mathrm{E}[R_2] = 0.14, & \sigma_2 = 0.40, & w_2 = 0.5, & \sigma_{1,3} = 0.04, \\ \mathrm{E}[R_3] = 0.20, & \sigma_3 = 0.80, & w_3 = 0.3, & \sigma_{2,3} = 0.10. \end{array}$$

We are able to represent the set of security returns with a single vector **r** and the set of portfolio weights with a single vector **w**:

$$\mathbf{r} = \begin{bmatrix} 0.12 \\ 0.14 \\ 0.20 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix}.$$

Bearing in mind that $\sigma_{i,j} = \sigma_{j,i}$ and that $\sigma_i^2 = \sigma_{i,i}$ (because the covariance between anything and itself equals variance), we may represent a covariance matrix for the securities as follows:

$$\mathbf{V} = \begin{bmatrix} 0.09 & 0.01 & 0.04 \\ 0.01 & 0.16 & 0.10 \\ 0.04 & 0.10 & 0.64 \end{bmatrix}.$$

Note that each element $v_{i,j}$ equals the covariance $\sigma_{i,j}$. A portfolio covariance matrix represents the set of security variances and covariances comprising the portfolio. The covariance matrix is always square and symmetric, with elements along the principal diagonal representing security variances. Each covariance between nonidentical securities is represented twice in the matrix. For example, the covariance between returns on security 1 and 2 is the same as the covariance between returns on securities 2 and 1. These covariances are represented by $v_{1,2} = v_{2,1} = 0.01$.

7.2 Matrix Arithmetic
(Background reading: sections 2.9 and 7.1)

Matrix arithmetic provides for standard rules of operation just as conventional arithmetic. Matrices may be added or subtracted if their dimensions are identical. Matrices **A** and **B** add to **C** if $a_{i,j} + b_{i,j} = c_{i,j}$ for all i and j:

$$\underset{\mathbf{A}}{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}} + \underset{\mathbf{B}}{\begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}} = \underset{\mathbf{C}.}{\begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,n} \end{bmatrix}}, \tag{7.1}$$

For example,

$$\underset{\mathbf{A}}{\begin{bmatrix} 2 & 4 & 9 \\ 6 & 4 & 25 \\ 0 & 2 & 11 \end{bmatrix}} + \underset{\mathbf{B}}{\begin{bmatrix} 3 & 0 & 6 \\ 2 & 1 & 3 \\ 7 & 0 & 4 \end{bmatrix}} = \underset{\mathbf{C}.}{\begin{bmatrix} 5 & 4 & 15 \\ 8 & 5 & 28 \\ 7 & 2 & 15 \end{bmatrix}},$$

Note that each of the three matrices is of dimension 3×3 and that each of the elements in matrix **C** is the sum of corresponding elements in matrices **A** and **B**. The process of subtracting matrices is similar, where $d_{i,j} - e_{i,j} = f_{i,j}$ for $\mathbf{D} - \mathbf{E} = \mathbf{F}$:

$$\begin{bmatrix} d_{1,1} & d_{1,2} & \dots & d_{1,n} \\ d_{2,1} & d_{2,2} & \dots & d_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{m,1} & d_{m,2} & \dots & d_{m,n} \end{bmatrix} - \begin{bmatrix} e_{1,1} & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} & \dots & e_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ e_{m,1} & e_{m,2} & \dots & e_{m,n} \end{bmatrix} = \begin{bmatrix} f_{1,1} & f_{1,2} & \dots & f_{1,n} \\ f_{2,1} & f_{2,2} & \dots & f_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{m,1} & f_{m,2} & \dots & f_{m,n} \end{bmatrix}, \tag{7.2}$$

$$\mathbf{D} \quad - \quad \mathbf{E} \quad = \quad \mathbf{F}.$$

For example,

$$\begin{bmatrix} 9 & 4 & 9 \\ 6 & 4 & 8 \\ 5 & 2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 6 \\ 2 & 1 & 6 \\ 5 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 3 \\ 4 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix},$$

$$\mathbf{D} \quad - \quad \mathbf{E} \quad = \quad \mathbf{F}.$$

Now consider a third matrix operation. The *transpose* $\mathbf{A}'$ of matrix $\mathbf{A}$ is obtained by interchanging the rows and columns of matrix $\mathbf{A}$. Each $a_{i,j}$ becomes $a_{j,i}$. The following represent matrix $\mathbf{A}$ and its transpose $\mathbf{A}'$:

$$\begin{bmatrix} 1 & 8 & 9 \\ 6 & 4 & 25 \\ 3 & 2 & 35 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 & 3 \\ 8 & 4 & 2 \\ 9 & 25 & 35 \end{bmatrix},$$

$$\mathbf{A} \qquad\qquad \mathbf{A}'.$$

The transpose of a column vector is a row vector:

$$\begin{bmatrix} 9 \\ 6 \\ 3 \\ 7 \end{bmatrix}, \qquad [9 \quad 6 \quad 3 \quad 7],$$

$$\mathbf{y} \qquad\qquad \mathbf{y}'.$$

Similarly, the transpose of a row vector is a column vector. Note that the transpose $\mathbf{V}'$ of a symmetric matrix $\mathbf{V}$ is $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} 0.09 & 0.01 & 0.04 \\ 0.01 & 0.16 & 0.10 \\ 0.04 & 0.10 & 0.64 \end{bmatrix}, \qquad \mathbf{V}' = \begin{bmatrix} 0.09 & 0.01 & 0.04 \\ 0.01 & 0.16 & 0.10 \\ 0.04 & 0.10 & 0.64 \end{bmatrix} = \mathbf{V}.$$

Matrix multiplication is somewhat more complex than matrix addition or subtraction. Furthermore, as we will discuss later, the order of matrices to be multiplied does affect the result. If two matrices may be multiplied, they are said to be *conformable* for multiplication. Any matrix may be multiplied by a scalar. One simply multiplies each of the elements of the matrix by the scalars to obtain the corresponding element of the product; that is, each element $c_{i,j}$ of **C** equals $sa_{i,j}$, where **C** = **sA**. More generally, matrices are said to conform for multiplication if the number of rows in the first matrix equals the number of rows in the second to be multiplied. That is, two matrices **A** and **B** may be multiplied to obtain the product **AB** = **C** if the number of columns in the first matrix **A** equals the number of rows **B** in the second. If matrix **A** is of dimension $m \times n$ and matrix **B** is of dimension $n \times q$, the dimensions of the product matrix **C** will be $m \times q$. Each element $c_{i,k}$ of matrix **C** is determined by the following sum:

$$c_{i,k} = \sum_{j=1}^{n} a_{i,j} b_{j,k}. \tag{7.3}$$

For example, consider the following product:

$$\begin{matrix} \begin{bmatrix} 5 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 7 \end{bmatrix} & \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} & = & \begin{bmatrix} 136 & 142 \\ 170 & 184 \\ 94 & 104 \end{bmatrix}, \\ \mathbf{A} \quad \cdot & \mathbf{B} & = & \mathbf{C}. \end{matrix}$$

Matrix **C** in the above is found as follows:

$$\begin{matrix} \begin{bmatrix} 5 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 7 \end{bmatrix} & \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} & = & \begin{bmatrix} 5\cdot 7+4\cdot 5+9\cdot 9 & 5\cdot 6+4\cdot 1+9\cdot 12 \\ 6\cdot 7+4\cdot 5+12\cdot 9 & 6\cdot 6+4\cdot 1+12\cdot 12 \\ 3\cdot 7+2\cdot 5+7\cdot 9 & 3\cdot 6+2\cdot 1+7\cdot 12 \end{bmatrix}, \\ \mathbf{A} \quad \cdot & \mathbf{B} & = & \mathbf{C}. \end{matrix}$$

Notice that the number of columns (3) in matrix **A** equals the number of rows in matrix **B**. Thus matrices **A** and **B** conform for multiplication. Also note that the number of rows in matrix **C** equals the number of rows in matrix **A**; the number of columns in **C** equals the number of columns in matrix **B**.

Matrix multiplication differs somewhat from ordinary multiplication in that the commutative property does not hold. This means that, in general, **AB** ≠ **BA**, and that the order of multiplication does matter. Thus, we should distinguish between *premultiplication* and *postmultiplication*. The first matrix in a product is said to premultiply the second; the second is said to postmultiply the first.

Consider the following system of equations:

$$y_1 = 7x_1 + 4x_2,$$
$$y_2 = 5x_1 + 9x_2.$$

This system can be represented as follows:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\mathbf{y} \quad = \quad \mathbf{C} \quad \cdot \ \mathbf{x}.$$

Notice that $\mathbf{x}$ with two rows conforms for premultiplication by $\mathbf{C}$ with two columns. The resultant vector $\mathbf{y}$ has two rows. When two matrices are multiplied, the order of the resulting matrix will equal the number of rows in the first matrix and the number of columns of the second.

APPLICATION 7.2: PORTFOLIO MATHEMATICS, PART II
(Background reading: application 7.1 and section 7.2)

In application 7.1, we represented the security returns, weights, variances, and covariances with a set of appropriate matrices. We will now perform arithmetic operations on these matrices to determine the expected return and variance of the portfolio. First, we obtain the portfolio's expected return $E[R_p] = \mathbf{w'r}$ as follows:

$$E[R_p] = [0.20 \quad 0.50 \quad 0.30] \begin{bmatrix} 0.12 \\ 0.14 \\ 0.20 \end{bmatrix} = 0.154,$$

$$E[R_p] = \qquad \mathbf{w'} \qquad\qquad \mathbf{r} \quad = E[R_p].$$

Note that we transposed the weights vector to make it conform for multiplication with the returns vector (the first matrix to multiply must have the same number of columns as the number of rows in the second matrix to multiply). Since our desired product is a single number (a 1×1 matrix), we want the first matrix to have one row and the second to have one column. This is why we premultiply by the transposed matrix. Next, we find the variance of returns for this portfolio $\sigma_p^2 = \mathbf{w'Vw}$ as follows:

$$\sigma_p^2 = [0.20 \quad 0.50 \quad 0.30] \begin{bmatrix} 0.09 & 0.01 & 0.04 \\ 0.01 & 0.16 & 0.10 \\ 0.04 & 0.10 & 0.64 \end{bmatrix} \begin{bmatrix} 0.20 \\ 0.50 \\ 0.30 \end{bmatrix} = 0.138,$$

$$\sigma_p^2 = \qquad \mathbf{w'} \qquad\qquad \mathbf{V} \qquad\qquad \mathbf{w}.$$

We obtain this product by multiplying from left to right (remember that the commutative property does not hold for multiplication of matrices), starting with $\mathbf{w'V}$:

$$\underset{\mathbf{w}'}{[0.20 \quad 0.50 \quad 0.30]}\underset{\mathbf{V}}{\begin{bmatrix} 0.09 & 0.01 & 0.04 \\ 0.01 & 0.16 & 0.10 \\ 0.04 & 0.10 & 0.64 \end{bmatrix}},$$

$$= \underset{\mathbf{w}'\mathbf{V}}{[0.018 + 0.005 + 0.012 \quad 0.002 + 0.08 + 0.03 \quad 0.008 + 0.05 + 0.192]}$$

$$= \underset{\mathbf{w}'\mathbf{V}}{[0.035 \quad 0.112 \quad 0.25]}.$$

We now multiply **w′V** by **w** to obtain the portfolio variance:

$$\underset{\mathbf{w}'\mathbf{V}}{[0.035 \quad 0.112 \quad 0.25]}\underset{\mathbf{w}}{\begin{bmatrix} 0.20 \\ 0.50 \\ 0.30 \end{bmatrix}} = 0.138 = \sigma_{\mathrm{p}}^2.$$

Note that the three matrices that we multiplied were of dimension 1×3, 3×3, and 3×1. Our desired result is a single number, or a 1×1 matrix. Therefore, the first matrix in our product should have one row and the last matrix in our product should have one column. To ensure conformability for multiplication, the number of columns in each matrix should be the same as the number of rows in the following matrix. Confirm the following for our three-security portfolio based on the above weights and covariances:

$$\sigma_{\mathrm{p}}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} = 0.138 = \mathbf{w}'\mathbf{V}\mathbf{w}.$$

This confirmation should verify that our portfolio variance result will be identical to that realized using the simple portfolio equations from chapter 6.

APPLICATION 7.3: PUT–CALL PARITY
(Background reading: section 7.2)

Stock options grant their owners the right (but not the obligation) to purchase or sell shares of specified stock known as the underlying stock at a specified "exercise" price. One type of stock option is a *call*, which grants its owner the right to purchase shares of an underlying stock at the exercise price before the expiration date of the call. The value of a call at its expiration is given by the maximum of either zero or the difference between the stock price at expiration and the exercise price of the call:

$$C_{\mathrm{T}} = \mathrm{MAX}[(S_{\mathrm{T}} - X), 0].$$

Thus, if the call is exercised at its expiration, its value is equal to the value of the underlying stock less the exercise price at which it can be purchased due to the option. If the stock price is lower than the exercise price of the option at its expiration, the call is discarded; its value is zero.

The second type of stock option is a *put*, which grants its owner the right to sell the underlying stock at a specified exercise price on or before its expiration date. The following is the payoff function for the put at expiration:

$$p_{\mathrm{T}} = \mathrm{MAX}[(X - S_{\mathrm{T}}), 0].$$

In this application, we discuss a simple model that expresses the value of a put relative to the value of a call with terms identical to that of the put. First, assume that there exists a European put and a European call (with a value of C_0) written on the same underlying stock, which currently has a value equal to S_{T}. Both options expire at time T and have an exercise price equal to X. The riskless return rate is r_{f}. Since the payoff function of the call at expiration is $C_{\mathrm{T}} = \mathrm{MAX}[S_{\mathrm{T}} - X, 0]$ and the payoff function for the put is $p_{\mathrm{T}} = \mathrm{MAX}[X - S_{\mathrm{T}}, 0]$, the following system describes the pricing of a put in terms of the underlying stock, exercise price of options, and the call with the same terms as the put:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = -\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} + \begin{bmatrix} X \\ X \\ \vdots \\ X \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix},$$

$$\mathbf{p} = -\mathbf{S} + \mathbf{X} + \mathbf{C},$$

$$\mathrm{MAX}(X - S, 0) = -S + X + \mathrm{MAX}(S - X, 0).$$

This *put–call parity* relation holds regardless of the number of potential outcomes in the state space. The securities need not span the outcome space for the put–call parity relation to hold. Consider the following numerical example where there are three potential stock prices, 60, 100, and 140, and a 105 exercise price for the options:

$$\begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix} = -\begin{bmatrix} 140 \\ 100 \\ 60 \end{bmatrix} + \begin{bmatrix} 105 \\ 105 \\ 105 \end{bmatrix} + \begin{bmatrix} 35 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{p} = -\mathbf{S} + \mathbf{X} + \mathbf{C}.$$

Because the put–call parity relation must hold at option expiry regardless of the underlying stock terminal price, the following put–call parity relation must hold at time zero:

$$p_0 = -S_0 + X\mathrm{e}^{-r_f T} + C_0.$$

That is, a put is equivalent in value to a portfolio consisting of a short position in one share of stock (the share is short sold; that is, borrowed and sold with the intent to repurchase at time T) underlying the put, an investment into a riskless asset certain to pay X at time T where X is the exercise price of both options and T is the term to option expiry, and a long position in one call on the stock with the same expiration date and exercise price as the put.

Suppose that the stock whose time T payoff function given above is currently selling for \$106. Further assume that a one-year call (let $T = 1$) with an exercise price equal to \$105 is currently selling for \$18 and the current riskless return rate equals 0.10. A one-year put with an exercise price equal to \$105 will be worth \$7.0079 according to the put–call parity relation:

$$7.0079 = -106 + 105e^{-0.1\cdot 1} + 18.$$

7.3 Inverting Matrices
(Background reading: sections 3.3 and 7.2)

For any real number $x \neq 0$, there exists a unique number x^{-1} such that $x\, x^{-1} = 1 = x^{-1}x$. The number x^{-1} is said to be the *multiplicative inverse* of x. For example, since $2^{-1} = \frac{1}{2}$, $2 \cdot 2^{-1} = 1$ and 2^{-1} is the inverse of 2. An *inverse* matrix $\mathbf{A}^{-1}$ exists for the square matrix $\mathbf{A}$ if the product $\mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{A}\mathbf{A}^{-1}$ equals the identity matrix $\mathbf{I}$. Consider the following product:

$$\underset{\mathbf{A}}{\begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix}} \underset{\mathbf{A}^{-1}}{\begin{bmatrix} -\frac{1}{30} & \frac{2}{15} \\ \frac{4}{15} & -\frac{1}{15} \end{bmatrix}} \underset{=}{=} \underset{\mathbf{I}.}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}},$$

If there exists a matrix $\mathbf{A}^{-1}$ which can be used to premultiply or postmultiply $\mathbf{A}$, matrix $\mathbf{A}$ is said to be *invertible*. Only square matrices may have inverses.[1] Those matrices that have inverses are said to be *nonsingular*; matrices that do not have inverses are said to be *singular*.

There are many computational techniques for finding the inverse matrix $\mathbf{A}^{-1}$ for matrix $\mathbf{A}$. The process that we will discuss here is the *Gauss–Jordan method*. There exist many other procedures for inverting matrices, but none are significantly more efficient and applicable generally than the Gauss–Jordan method for matrices of higher order than 3. The Gauss–Jordan method will be performed on matrix $\mathbf{A}$ by first augmenting it with the identity matrix as follows:

$$\left[\begin{array}{cc:cc} 2 & 4 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right]. \tag{A}$$

We will refer to the augmented matrix as matrix $\mathbf{A}$. Next, a series of row operations (see 1, 2, and 3 below) will be performed until the identity matrix appears on the left

[1] We will ignore left inverses and right inverses, which may exist for matrices that are not square.

QRMC07 9/17/01 4:45 PM Page 124

side of matrix **A**, replacing the original elements in the left side. Since the same row operations will be performed on the left side, the right side elements will comprise the inverse matrix $\mathbf{A}^{-1}$. Thus, in our final augmented matrix, we will have ones along the principal diagonal on the left side and zeros elsewhere; the right side of the matrix will comprise the inverse of **A**. Allowable row operations include the following:

1 Multiply a given row by any nonzero constant. Each element in the row must be multiplied by the same constant.
2 Add a given row to any other row in the matrix. Each element in a row is added to the corresponding element in the same column of another row.
3 Any combination of the above. For example, a row may be multiplied by a negative constant before it is added to another row.

We will perform a very systematic series of row operations, where we first obtain 1 for element *j* in row *j* and then zeros elsewhere in column *j*. Our first row operation will serve to replace the upper left corner value 2 with 1. We multiply row 1 in **A** (row 1**A**) by 0.5 to obtain the following:

$$\left[\begin{array}{cc:cc} 1 & 2 & 0.5 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right], \qquad 1\mathbf{A} \cdot 0.5 = 1\mathbf{B},$$

where row 1**B** replaces row 1**A**. Now we obtain a zero in the lower left corner by multiplying row 2 in **A** by $\frac{1}{4}$ and subtracting the result from our new row 1 to obtain matrix **B** as follows:

$$\left[\begin{array}{cc:cc} 1 & 2 & 0.5 & 0 \\ 0 & 1.5 & 0.5 & -\frac{1}{4} \end{array}\right], \qquad \begin{array}{l} 1\mathbf{A} \cdot 0.5 = 1\mathbf{B} \\ 1\mathbf{B} - (2\mathbf{A} \cdot \frac{1}{4}) = 2\mathbf{B} \end{array}. \tag{B}$$

Next, we move to the second row. We obtain a 1 in the lower right corner of the left side of the matrix by multiplying row 2**B** by $\frac{2}{3}$:

$$\left[\begin{array}{cc:cc} 1 & 2 & 0.5 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{6} \end{array}\right], \qquad 2\mathbf{B} \cdot (-\tfrac{2}{3}) = 2\mathbf{C}.$$

Notice that we obtained this lower right corner value of 1 without affecting the 1 or 0 that we previously obtained in matrix **B**. We obtain a zero in the upper right corner of the left side matrix by multiplying row 2 above by 2 and subtracting from row 1 in matrix **B**:

$$\left[\begin{array}{cc:cc} 1 & 0 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{6} \end{array}\right], \qquad \begin{array}{l} 1\mathbf{B} - (2\mathbf{C} \cdot 2) = 1\mathbf{C} \\ 2\mathbf{B} \cdot (-\frac{2}{3}) = 2\mathbf{C} \end{array}. \tag{C}$$

The left side of augmented matrix **C** is the identity matrix and the and the right side of **C** is $\mathbf{A}^{-1}$, the inverse of matrix **A**.

QRMC07 9/17/01 4:45 PM Page 125

Matrices cannot be divided as numbers can be in arithmetic. However, since one can invert a matrix, this inverted matrix can be premultiplied by another matrix to obtain a result analogous to a quotient in dividing. This process of inverting matrices is most useful for solving many types of linear equations.

7.4 Solving Systems of Linear Equations
(Background reading: sections 1.3, 3.3, and 7.3)

Matrices are very useful for arranging systems of equations for repetitive calculations. Solving systems of linear equations simultaneously for variables is a particularly useful application for matrix inverses. For example, consider the following system:

$$0.1x_1 + 0.24x_2 = 0.1,$$
$$0.2x_1 + 0.6x_2 = 0.16.$$

In matrix format, this system is represented as follows:

$$\underset{\mathbf{C}}{\begin{bmatrix} 0.1 & 0.24 \\ 0.2 & 0.6 \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \underset{=}{=} \underset{\mathbf{s}.}{\begin{bmatrix} 0.1 \\ 0.16 \end{bmatrix}},$$

We are not able to divide $\mathbf{s}$ by $\mathbf{C}$ to obtain $\mathbf{x}$; instead, we invert $\mathbf{C}$ to obtain $\mathbf{C}^{-1}$ and postmultiply it by $\mathbf{s}$ to obtain $\mathbf{x}$:

$$\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}$$

Therefore, to solve for vector $\mathbf{x}$, we first invert $\mathbf{C}$ by augmenting it with the identity matrix:

$$\left[\begin{array}{cc:cc} 0.1 & 0.24 & 1 & 0 \\ 0.2 & 0.6 & 0 & 1 \end{array}\right]. \tag{A}$$

We then perform a series of row operations to invert matrix $\mathbf{C}$ as follows:

$$\left[\begin{array}{cc:cc} 1 & 2.4 & 10 & 0 \\ 0 & 0.6 & -10 & 5 \end{array}\right], \quad \begin{array}{l} \text{row } 1\mathbf{B} = 1\mathbf{A} \cdot 10 \\ \text{row } 2\mathbf{B} = (5 \cdot 2\mathbf{A}) - 1\mathbf{B} \end{array}; \tag{B}$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 50 & -20 \\ 0 & 1 & -\frac{50}{3} & \frac{25}{3} \end{array}\right], \quad \begin{array}{l} \text{row } 1\mathbf{C} = 1\mathbf{B} - (2.4 \cdot 2\mathbf{C}) \\ \text{row } 2\mathbf{C} = 2\mathbf{B} \cdot \frac{5}{3} \end{array}; \tag{C}$$

$$\mathbf{I} \qquad \mathbf{C}^{-1}.$$

QRMC07 9/17/01 4:45 PM Page 126

Thus, we obtain vector **x** with the following product:

$$\underset{\mathbf{C}^{-1}}{\begin{bmatrix} 50 & 20 \\ -\frac{200}{3} & \frac{100}{3} \end{bmatrix}} \cdot \underset{\mathbf{s}}{\begin{bmatrix} 0.1 \\ 0.16 \end{bmatrix}} = \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\mathbf{x}.}{\begin{bmatrix} 1.8 \\ -\frac{1}{3} \end{bmatrix}}, \tag{D}$$

Thus, we find that $x_1 = 1.8$ and $x_2 = -\frac{1}{3}$.

APPLICATION 7.4: EXTERNAL FUNDING REQUIREMENTS (Background reading: section 7.5 and application 3.7)

Recall the Albert Company whose financial statements are given in tables 3.1 and 3.2, in application 3.7. The firm's Earnings Before Interest and Tax ($EBIT$) level was projected to be \$300,000 next year. The firm had previously borrowed \$600,000, requiring \$50,000 in interest payments. Management expected the firm to remain in the 40% corporate income tax bracket ($\tau = 0.4$) and pay out one third of its after-tax earnings in dividends ($\delta = 0.333$). Since the firm's production level is expected to increase next year, management has determined that each asset account must also increase by 40%. Assets currently total \$1,000,000 and will increase to \$1,400,000. Current liabilities will also increase from its present level of \$150,000 by 40% or ΔCL = \$60,000. The firm pays no interest on its current liabilities. Managers have already decided to sell bonds at an interest rate of 10% to provide any external capital necessary to finance the asset level increase. Management's problem is to determine how much additional capital to raise through this 10% bond issue. Based on this information, we should be able to determine the Albert Company's external financing needs (EFN) for next year.

Since management has determined that it must change its asset total by $\Delta Assets$ = \$400,000, it must determine how these assets will be financed. That is, management must determine the total sum of capital required to support the change in the total asset level. Some of this necessary capital can be derived from internal sources such as retained earnings (RE) or changes in current liabilities (ΔCL). These sources are likely to change simultaneously with the firm's production level and provide capital directly from the increase in the firm's level of operation. For example, an increase in the firm's sales level may result directly in an increase in the firm's level of retained earnings, since revenues, variable costs, and profits can be expected to increase. Furthermore, as the firm's sales level increases, it may be reasonable to anticipate an increase in the firm's number of employees, further resulting in an increase in the firm's accrued wages level. Other current liability levels are likely to increase in a similar manner. The remaining funds must be obtained through some external source such as the sale of long-term bonds or equity. In summary, the amount of money the firm must raise from external sources is determined by the following equation:

$$EFN = \Delta Assets - \Delta CL - RE = \$400{,}000 - \$60{,}000 - RE = \$340{,}000 - RE,$$

where

$$RE = (EBIT - INT) \cdot (1 - \tau) \cdot (1 - \delta)$$
$$= (\$300{,}000 - \$50{,}000 - i \cdot EFN) \cdot (1 - 0.4) \cdot (1 - 0.333),$$

which can be simplified as follows:

$$RE = (\$250{,}000 \cdot 0.4) - 0.04 \cdot EFN = \$100{,}000 - 0.04 \cdot EFN.$$

Our EFN and RE functions above can be rewritten as

$$\$100{,}000 = RE + 0.04 \cdot EFN,$$
$$\$340{,}000 = RE + EFN.$$

We now have two equations with two variables to solve. This problem is structured in matrix format as follows:

$$\begin{bmatrix} 1 & 0.04 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} RE \\ EFN \end{bmatrix} = \begin{bmatrix} 100{,}000 \\ 340{,}000 \end{bmatrix}$$
$$\mathbf{C} \quad \cdot \quad \mathbf{x} \quad = \quad \mathbf{s}.$$

Performing a series of row operations to obtain matrix $\mathbf{C}^{-1}$ yields the following:

$$\begin{bmatrix} 1.041667 & -0.04167 \\ -1.04167 & 1.041667 \end{bmatrix} \begin{bmatrix} 100{,}000 \\ 340{,}000 \end{bmatrix} = \begin{bmatrix} 90{,}000 \\ 250{,}000 \end{bmatrix} = \begin{bmatrix} RE \\ EFN \end{bmatrix},$$
$$\mathbf{C}^{-1} \quad \cdot \quad \mathbf{s} \quad = \quad \mathbf{x} \quad = \quad \mathbf{x}.$$

Thus, $RE = \$90{,}000$ and $EFN = \$250{,}000$. More generally, the following equation may be used to solve simultaneously for RE and EFN:

$$\begin{bmatrix} 1 & [i \cdot (1 - \tau) \cdot (1 - \delta)] \\ 1 & 1 \end{bmatrix} \begin{bmatrix} RE \\ EFN \end{bmatrix} = \begin{bmatrix} [(EBIT - INT_{EX}) \cdot (1 - \tau) \cdot (1 - \delta)] \\ (\Delta Assets - \Delta CL) \end{bmatrix},$$
$$\mathbf{C} \quad \cdot \quad \mathbf{x} \quad = \quad \mathbf{s}.$$

where INT_{EX} represents the interest payments on existing debt and i represents the interest rate on new debt. If equity rather than debt is to finance the asset increase, zero may be substituted for i and the system solves rather easily. In any case, solving this system requires substituting values in for variables, inverting matrix $\mathbf{C}$, and solving for vector $\mathbf{x}$.

APPLICATION 7.5: COUPON BONDS AND DERIVING YIELD CURVES (Background reading: sections 4.8, 4.9, and 7.4 and applications 2.6 and 4.4)

The *spot rate* is the spot rate at present prevailing for zero coupon bonds of a given maturity. The t-year yield $y_{0,t}$ of a zero coupon bond is determined as follows:

$$y_{0,t} = \sqrt[t]{\frac{F}{P_0}} - 1,$$

where F is the face value or principal of the bond, P_0 is the purchase price of the bond, and t is the zero coupon bond's term to maturity. The t-year spot rate is denoted here by $y_{0,t}$, which represents an implied spot rate on a loan to be made at time zero and repaid in its entirety at time t. Spot rates may be estimated from bonds with known future cash flows and their current prices. We are able to obtain spot rates from yields implied from series of bonds when we assume that the Law of One Price holds. Recall that the Law of One Price maintains that securities generating identical cash flows must sell for the same price.

The yield curve represents yields or spot rates of bonds with varying terms to maturity. For example, at a given moment in time, the yield or spot rate for one-year bonds may be 4% ($y_{0,1} = 0.04$), while the yield for five-year bonds may be 6% ($y_{0,5} = 0.06$). This section is concerned with how interest rates or yields vary with maturities of bonds. The simplest bonds to work with from an arithmetic perspective are *pure discount notes*, also known as zero coupon notes, which make no interest payments. Such notes make only one payment at one point in time – on the maturity date of the note. Determining the relationship between yield and term to maturity for these bonds is quite trivial. The return that one obtains from a pure discount note is strictly a function of capital gains; that is, the difference between the face value F of the note and its purchase price P_0. Short-term U.S. Treasury bills are an example of pure discount (or zero coupon) notes. Coupon bonds are somewhat more difficult to work with from an arithmetic perspective because they make payments to bondholders at a variety of different periods. Since they make multiple payments, coupon bonds are analogous to portfolios of pure discount bonds.

A coupon bond may be treated as a portfolio of pure discount notes, with each coupon being treated as a separate note maturing on the date the coupon is paid. Using coupon bonds slightly complicates the process for determining yields, but is necessary when there aren't pure discount notes maturing in key time periods. Consider an example involving three bonds whose terms and prices are given in table 7.1. The three bonds are trading at known prices with a total of eight annual coupon payments among them (two for bond A and three each for bonds B and C). Bond yields or spot rates must be determined simultaneously to avoid associating contradictory rates for the annual coupons on each of the three bills.

Table 7.1 Coupon bonds A, B, and C

Bond	Current price	Face value	Coupon rate	Years to maturity
A	947.376	1,000	0.06	2
B	904.438	1,000	0.08	3
C	980.999	1,000	0.10	3

Let D_t be the discount function for time t; that is, $D_t = 1/(1 + y_{0,t})^t$. Since $y_{0,t}$ is the spot rate or discount rate that equates the present value of a bond with its current price, the following equations may be solved for discount functions then spot rates:

$$947.376 = 50D_1 + 1{,}050D_2,$$
$$904.438 = 60D_1 + 60D_2 + 1{,}060D_3,$$
$$980.999 = 90D_1 + 90D_2 + 1{,}090D_3.$$

This system of equations may be represented by the following system of matrices:

$$\begin{bmatrix} 50 & 1{,}050 & 0 \\ 60 & 60 & 1{,}060 \\ 90 & 90 & 1{,}090 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 947.376 \\ 904.438 \\ 980.999 \end{bmatrix},$$
$$\mathbf{CF} \cdot \mathbf{d} = \mathbf{P}_0.$$

To solve this system we first invert matrix **CF** to obtain $\mathbf{CF}^{-1}$. We then use this inverse matrix to premultiply vector $\mathbf{P}_0$ to obtain vector **d**:

$$\begin{bmatrix} -0.001 & -0.03815 & 0.0371 \\ 0.001 & 0.001817 & -0.00177 \\ 0 & 0.003 & -0.002 \end{bmatrix} \begin{bmatrix} 947.376 \\ 904.438 \\ 980.999 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0.943396 \\ 0.857339 \\ 0.751315 \end{bmatrix},$$
$$\mathbf{CF}^{-1} \cdot \mathbf{P}_0 = \mathbf{d}.$$

Thus, we find from solving this system for vector **d** that $D_1 = 0.943396$, $D_2 = 0.857339$, and $D_3 = 0.751315$. Since $D_t = 1/(1 + y_{0,t})^t$, $1/D_t = (1 + y_{0,t})^t$, and $y_{0,t} = 1/D^{1/t} - 1$, spot rates are determined as follows:

$$\frac{1}{D_1} - 1 = \frac{1}{0.943396} - 1 = 0.06,$$
$$\frac{1}{D_2^{1/2}} - 1 = \frac{1}{0.857339^{1/2}} - 1 = 0.08,$$
$$\frac{1}{D_3^{1/3}} - 1 = \frac{1}{0.751315^{1/3}} - 1 = 0.10.$$

In this example, there exists a different spot rate (or discount rate) for each term to maturity. However, the spot rates for all cash flows generated by all bonds at a given period of time are the same. This consistency is necessary for the market to avoid arbitrage opportunities. Thus, $y_{0,t}$ will vary over terms to maturity, but all bonds in the market will be subject to this single yield for a given time period.

APPLICATION 7.6: ARBITRAGE WITH RISKLESS BONDS (Background reading: application 7.5)

Arbitrage was defined in section 1.4 as the simultaneous purchase and sale of assets or portfolios yielding identical cash flows. Assets generating identical cash flows (certain or risky cash flows) should be worth the same amount. This is known as the Law of One Price. If assets generating identical cash flows sell at different prices, opportunities exist to create a profit by buying the cheaper asset or combination and selling the more expensive asset or combination. The ability to realize a profit from this type of transaction is known as an arbitrage opportunity. Solutions for multiple variables in systems of equations are most useful in the application of the Law of One Price and for those seeking arbitrage opportunities.

The example given in application 7.5 above consisted of three default risk-free bonds. The result defined spot rates for all three relevant years. The cash flow structure of any one-, two-, or three-year bond that might be added to this market can be replicated by some portfolio of bonds A, B, and C. Consider for example, Bond D, a three-year issue that can be replicated by a portfolio of bonds A, B, and C. Assume for this example that there now exists Bond D, a three-year, 12% coupon issue selling in this market for \$1,040. This bond will make payments of \$120 in years 1 and 2 followed by a \$1,120 payment in year 3. A portfolio of bonds A, B, and C can be comprised to exactly replicate this cash flow series. Thus, we will determine exactly how much to invest in each of our bonds A, B, and C to replicate Bond D. Portfolio weights to replicate Bond D will be determined by the following system of equations or matrices:

$$\begin{aligned}
120 &= 50w_A + 60w_B + 90w_C,\\
120 &= 1{,}050w_A + 60w_B + 90w_C,\\
1{,}120 &= 0w_A + 1{,}060w_B + 1{,}090w_C;
\end{aligned}$$

$$\begin{bmatrix} 50 & 60 & 90 \\ 1{,}050 & 60 & 90 \\ 0 & 1{,}060 & 1{,}090 \end{bmatrix}\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} 120 \\ 120 \\ 1{,}120 \end{bmatrix},$$

$$\mathbf{CF} \cdot \mathbf{w} = \mathbf{cf}_D.$$

To solve this system, we first invert matrix **CF**, then use it to premultiply vector $\mathbf{cf}_D$ to obtain vector **w**:

$$\begin{bmatrix} -0.001 & 0.001 & 0 \\ -0.03815 & 0.001817 & 0.003 \\ 0.0371 & -0.00177 & -0.002 \end{bmatrix}\begin{bmatrix} 120 \\ 120 \\ 1{,}120 \end{bmatrix} = \begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix},$$

$$\mathbf{CF}^{-1} \cdot \mathbf{cf}_D = \mathbf{w}.$$

Thus, we find from this system that $w_A = 0$, $w_B = -1$, and $w_C = 2$. We determine the value of the portfolio replicating bond D by weighting the current market prices of

QRMC07 9/17/01 4:45 PM Page 131

bonds B and C from application 7.5: (–1 · \$904.438) + (2 · \$980.999) = \$1,057.56. This means that we short sell (borrow and then sell) one bond B and purchase two of bond C.[2] Cash flows in years one, two, and three are given as follows:

Year	Bond B	Bond C	Portfolio	Bond D
1	–60	180	120	120
2	–60	180	120	120
3	–1,060	2,180	1,120	1,120

Thus, the cash flows from the portfolio exactly match the cash flows generated by bond D. Thus, by the Law of One Price, bond D should sell for the same price as the portfolio. Based on the portfolio's price, the value of bond D was calculated to be \$1,057.56, although its actual current market price is \$1,040. Thus, one gains an arbitrage profit from the purchase of this bond for \$1,040 financed by the sale of the portfolio containing the short position in bond B (one B is now purchased since the portfolio is to be sold) and two bonds C for a total price of \$1,057.56 for the portfolio. Here, we simply swap a portfolio comprised of bonds B and C for bond D. Our cash flows in years 1, 2, and 3 will be zero, although we receive a positive cash flow at time zero of \$17.56 (–\$904.438 + 2 · \$980.999 – \$1,040). This is a clear arbitrage profit. Again, to realize this \$17.56 arbitrage profit, the arbitrageur purchases one bond B, sells two bonds C, and buys one bond D. This arbitrage opportunity will persist until the value of the portfolio equals the value of bond D.

APPLICATION 7.7: FIXED INCOME PORTFOLIO DEDICATION (Background reading: application 7.6)

A fixed income portfolio is concerned with assuring the provision of a relatively stable income over a period of time. Typically, a fixed income fund is expected to provide a fixed series of payments to its creditors, clients, or owners for a given period. For example, a pension fund is often expected to make a series of fixed payments to pension fund participants. Such funds must invest their assets to ensure that their liabilities are paid. In many cases, fixed income funds will purchase assets such that their cash inflows exactly match the liability payments that they are required to make. This exact matching strategy is referred to as portfolio dedication; that is, cash flows generated by assets are dedicated to assuring payments required by creditors or other stakeholders. Portfolio dedication is intended to minimize the risk of the fund. The process of dedication is quite similar to the arbitrage swaps discussed in application 7.6 above. The fund manager determines the cash flows associated with his liability structure and replicates the cash flow structure with a series of default risk-free bonds. For example, assume

[2] Short selling a bond occurs when one borrows a bond from another investor and sells it on his own account. The short-seller needs to replace the interest payments in the lender's account as well as the principal when the borrowed bond matures. One short sells to satisfy a hedging or portfolio strategy or when the bond's price is expected to drop.

Table 7.2 Coupon bonds E, F, and G

Bond	Current price	Face value	Coupon rate	Years to maturity
E	1,000	1,000	0.10	2
F	1,035	1,000	0.12	2
G	980	1,000	0.10	3

that a pension fund manager needs to make payments to pension plan participants of \$10,000,000 in one year, \$20,000,000 in two years, and \$30,000,000 in three years. She needs to match these cash flows with a portfolio of bonds E, F, and G, whose characteristics are given in table 7.2. These three bonds must be used to match the cash flows associated with the fund's liability structure. For example, in year 1, bond E will pay \$100, F will pay \$120, and G will pay \$100. These payments must be combined to total \$10,000,000. Cash flows must be matched in years 2 and 3 as well. Only one matching strategy exists for this scenario. The following system may be solved for **b** to determine exactly how many of each of the bonds are required to satisfy the fund's cash flow requirements:

$$\begin{bmatrix} 100 & 120 & 100 \\ 1{,}100 & 1{,}120 & 100 \\ 0 & 0 & 1{,}100 \end{bmatrix} \begin{bmatrix} b_{\text{E}} \\ b_{\text{F}} \\ b_{\text{G}} \end{bmatrix} = \begin{bmatrix} 10{,}000{,}000 \\ 20{,}000{,}000 \\ 30{,}000{,}000 \end{bmatrix},$$

$$\mathbf{CF} \cdot \mathbf{b} = \mathbf{L}.$$

Inverting matrix **CF** and multiplying by vector **L** yields the following system:

$$\begin{bmatrix} -0.056 & 0.006 & 0.00454545 \\ 0.055 & -0.005 & -0.00454545 \\ 0 & 0 & 0.000909091 \end{bmatrix} \begin{bmatrix} 10{,}000{,}000 \\ 20{,}000{,}000 \\ 30{,}000{,}000 \end{bmatrix} = \begin{bmatrix} -303{,}636.36 \\ 313{,}636.36 \\ 27{,}272.72 \end{bmatrix},$$

$$\mathbf{CF}^{-1} \cdot \mathbf{L} = \mathbf{b}.$$

Thus, we find that the sale of 303,636.36 bonds E, and the purchase of 313,636.36 bonds F and 27,272.72 bonds G satisfies the manager's exact matching requirements. The fund's time zero payment for these bonds totals \$47,704,545 at their current market prices.

APPLICATION 7.8: BINOMIAL OPTION PRICING
(Background reading: section 7.4 and application 7.3)

Consider a one-time-period, two-potential-outcome framework where there exists company X stock currently selling for \$100 per share and a riskless \$100 face-value T-bill currently selling for \$90. Suppose that company Q faces uncertainty, such that it will pay its owner either \$60 or \$140 in one year. The T-bill will certainly pay its owner \$100 in one year. Further assume that a call with an exercise price of \$110 exists on one share of Q stock. This call will be worth either \$0 or \$30 when it expires, based on the value of the underlying stock. More generally, the value of the call at expiration is the larger of either zero or the difference between the value of its underlying security and the call exercise price. In this example, if the stock is worth \$60, the call is worthless; if the stock is worth \$140, the call is worth \$30. The payoff vectors for stock q, the T-bill (**b**), and the call (**c**) are given as follows:

$$\mathbf{x} = \begin{bmatrix} 60 \\ 140 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 0 \\ 30 \end{bmatrix}.$$

Since the payoff vectors of the stock and T-bill span the two-outcome space in this two-potential-outcome framework, they form the basis for the two-dimensional space. Therefore, the payoff structure for the call can be replicated by a portfolio consisting of shares of the underlying stock and T-bills as follows:

$$\begin{bmatrix} 60 & 100 \\ 140 & 100 \end{bmatrix} \begin{bmatrix} \#x \\ \#b \end{bmatrix} = \begin{bmatrix} 0 \\ 30 \end{bmatrix}.$$

To determine the number of shares and T-bills needed to replicate the call, we invert the payoffs matrix, to obtain:

$$\begin{bmatrix} -0.0125 & 0.0125 \\ 0.0175 & -0.0075 \end{bmatrix} \begin{bmatrix} 0 \\ 30 \end{bmatrix} = \begin{bmatrix} 0.375 \\ -0.225 \end{bmatrix}.$$

We find that $\#x = 0.375$ and $\#b = -0.225$. This implies that the payoff structure of a single call can be replicated with a portfolio comprising 0.375 shares of X company stock for a total of $0.375 \cdot \$100 = \$37.50$ and short-selling 0.225 T-bills (in effect, borrowing $0.225 \cdot \$90 = \$20.25$ at the T-bill rate). This portfolio requires a net investment of $\$37.50 - \$20.25 = \$17.00$. Since the call on X company stock has the same payoff structure as this portfolio, its current value must be \$17.

7.5 Spanning the State Space (Background reading: sections 7.3 and 7.4)

A vector is a matrix with either only one row or one column. Any column vector $\mathbf{v}$ consisting of n real elements is said to be within the set $\mathbb{R}^n$, which represents the n-dimensional *vector space*. $\mathbb{R}^n$ is defined as the set of all vectors with n real-valued elements or coordinates. The following represent vectors in three-dimensional space:

$$\mathbf{a} = \begin{bmatrix} 4 \\ 2 \\ 9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -5 \\ 14 \\ 22 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 10 \\ 10 \\ -3 \end{bmatrix}; \qquad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3.$$

Vectors **a**, **b**, and **c** are all elements of the three-dimensional space $\mathbb{R}^3$ because each of the vectors contains three elements. Each of the n elements of a vector **v** might be regarded as a coordinate of a point in n-dimensional space or an n-dimensional hyperplane. All of the vectors falling within this hyperplane are said to exist in set $\mathbb{R}^n$.

Each of these three vectors might be said to represent cash flows for a security over a three-year time period, as in applications 7.5–7.7 above. Alternatively, the three elements in any one of these vectors might represent cash flows contingent on potential outcomes of an uncertain scenario, as in applications 7.3 and 7.8 above. Hence, each of the dimensions in an n-dimensional vector space might represent cash flows over n periods or potential cash flows contingent on n possible outcomes. Cash flow vectors might also be structured to represent the combination of all potential cash flow outcomes in each of many time periods.

Vector addition and scalar multiplication are allowable vector operations. *Linear combinations* of given vectors are applications of these two vector operations. A linear combination of vectors may be executed by performing any one or combination of the following:

1 Multiplication of any vector by a scalar.
2 Addition of any combination of vectors either before or after multiplication by scalars.

If a vector in a given n-dimensional space can be expressed as a linear combination of a set of other vectors in the same space, we say that the given vector is *linearly dependent* on that set of vectors. Similarly, if a set of vectors $\{\mathbf{x}\}$ can be multiplied by a series of scalars α (where at least one of the scalars is nonzero) to obtain a vector of zeros, we say that linear dependence exists among the set of vectors $\{\mathbf{x}\}$:

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \ldots + \alpha_n\mathbf{x}_n = [0]. \tag{7.4}$$

If at least one of the scalars α_i above has a nonzero value, this set of n vectors **x** is said to be linearly dependent. *Linear independence* within a set of vectors $\{\mathbf{x}\}$ exists where the only set of scalars that satisfies this equality consists only of zeros. When linear independence exists for a set $\{\mathbf{x}\}$ of vectors, no vector in the set can be expressed as a linear combination of other vectors in the set.

Linear dependence exists within vector sets (A) and (B) below because equation (7.4) can be satisfied with scalars such that at least one $\alpha \neq 0$. Furthermore, within each set of vectors, any one vector may be described as a linear combination of the other two.

$$\begin{array}{ccc} \begin{bmatrix} 9 \\ 4 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{array} \qquad \begin{array}{l} 0\mathbf{x}_1 + 0.2\mathbf{x}_2 - 1\mathbf{x}_3 = [0], \\ 0\mathbf{x}_1 + 0.2\mathbf{x}_2 = \mathbf{x}_3; \end{array} \tag{A}$$

QRMC07 9/17/01 4:45 PM Page 135

$$\underset{\mathbf{x}_1}{\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}} \quad \underset{\mathbf{x}_2}{\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}} \quad \underset{\mathbf{x}_3}{\begin{bmatrix} 10 \\ 11 \\ 21 \end{bmatrix}} \qquad \begin{aligned} 1\mathbf{x}_1 + 2\mathbf{x}_2 - 1\mathbf{x}_3 &= [0], \\ 1\mathbf{x}_1 + 2\mathbf{x}_2 &= \mathbf{x}_3. \end{aligned} \tag{B}$$

We determine that vector set (A) is linearly dependent by demonstrating that there exists a set of values for α satisfying

$$\begin{aligned} 9\alpha_1 + 0\alpha_2 + 0\alpha_3 &= 0, \\ 4\alpha_1 + 5\alpha_2 + 1\alpha_3 &= 0, \\ 1\alpha_1 + 10\alpha_2 + 2\alpha_3 &= 0. \end{aligned}$$

It is obvious from the first equation that α_1 equals zero. Using α_1 equal to zero in the second and third equations, we find that any value for α_2 will satisfy the equality as long as α_3 equals $-5\alpha_2$. Since at least one of the scalars α may be nonzero when satisfying the three equations, this set of three vectors is linearly dependent. If it had been true that three nonzero scalars could not be found to satisfy the equations, linear independence would have existed within the set.

Each of vector sets (C) and (D) below are linearly independent because the set of scalars satisfying equation (7.4) for each set will all have zero values. For example, in vector set (C), α_1, α_2, and α_3 must all equal zero for a vector of zeros to be a linear combination of the three vectors. Furthermore, no vector in set (C) can be defined as a linear combination of the other vectors in set (C) and no vector in set (D) can be defined as a linear combination of the other vectors in set (E).

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Linearly independent vectors,} \tag{C}$$

$$\begin{bmatrix} 20 \\ 20 \\ 1{,}020 \end{bmatrix} \quad \begin{bmatrix} 45 \\ 45 \\ 1{,}045 \end{bmatrix} \quad \begin{bmatrix} 100 \\ 1{,}100 \\ 0 \end{bmatrix} \qquad \text{Linearly independent vectors.} \tag{D}$$

We will discuss how to verify whether linear independence exists shortly.

A set of n vectors is said to *span* the n-dimensional vector space if that set of n vectors is linearly independent. A set of n vectors consisting of n elements each, which do not span a vector space, are not linearly independent. Vector sets (A) and (B) do not span the n-dimensional vector space; vector sets (C) and (D) do span the n-dimensional vector space.

When a set of n vectors spans the n-dimensional vector space, this set of n vectors is known as the *basis* for the n-dimensional space. Any n-dimensional vector outside of this basis can be expressed as a linear combination of vectors in this basis. The basis for $\mathbb{R}^n$ is said to *span* the n-dimensional space. For example, any three-dimensional

QRMC07 9/17/01 4:45 PM Page 136

vector (such as [1,2,3]′, which equals $1\mathbf{x}_1 + 2\mathbf{x}_2 + 3\mathbf{x}_3$) can be expressed as a linear combination of the three vectors in set (C) above, since the three vectors in set (C) each have three elements and are linearly independent. This set of three vectors is a basis for $\mathbb{R}^3$; these three vectors span $\mathbb{R}^3$. The same can be said for vector set (D); this set represents a basis for and spans $\mathbb{R}^3$.

In a sense, when one vector is linearly dependent on another $n-1$ vectors, the information in the other $n-1$ vectors can be used to replicate the information in the nth vector. In a financial sense, where elements within a vector represent payoffs of a given security contingent on outcomes or associated with a points in time, the payoff structure of the nth security can be replicated with a portfolio comprising the other $n-1$ securities on which its payoff vector is linearly dependent. When a set of n payoff vectors span the n potential outcome or time space, the payoff structure for any other security or portfolio in the same outcome or time space can be replicated with the payoff vectors of the n securities forming the basis for the n-dimensional payoff space. Securities or portfolios whose payoff vectors can be replicated by portfolios of other securities must sell for the same price as those portfolios; otherwise, the Law of One Price is violated.

Consider application 7.6 above examining arbitrage opportunities in bond markets. An arbitrage opportunity will exist when the payoff vector of a given bond can be replicated by a linear combination of payoff vectors for other bonds, yet sell for a price different from the portfolio which replicates it. Thus, an arbitrage opportunity exists. In the absence of arbitrage opportunities, if that bond's payoff vector is linearly dependent on the payoff vectors of the $n-1$ bonds in the portfolio, the bond cannot sell for a price different from that of the portfolio. Application 7.7 concerning portfolio dedication also requires that the liability structure be dependent on the payoff vectors of bonds. In application 7.5, spot rates are implied based on the assumption that arbitrage opportunities do not exist among the bonds used to construct the yield curve. If one wishes to compute n spot rates, a minimum of n bonds forming the basis for the n-dimensional time space are required.

APPLICATION 7.9: USING OPTIONS TO SPAN THE STATE SPACE (Background reading: section 7.5 and application 7.8)

If a set of payoff vectors form the basis for the n-dimensional state space for an economy, any security that exists in that economy can be priced as a linear combination of those payoff vectors. This means that any security in an economy can be priced if n other securities with linearly independent payoff vectors are priced. In an actual setting, the difficulty with pricing securities with this methodology is locating securities whose payoff vectors can be exactly specified outcome-by-outcome. For example, how does one determine the exact element to place in the third row of payoff vectors for three unrelated stocks? Can we really identify what the payoff for each of these three stocks is in the third outcome?

Since derivative securities have payoff vectors that are contingent on the payoff vectors for other securities, one can define the outcome space relative to the payoff vector for the underlying security. One can create unlimited numbers of derivative securities

QRMC07 9/17/01 4:45 PM Page 137

such as options on stocks or other existing assets. Payoff vectors for these options can be linearly independent. The state space can be spanned with the underlying security and the options written on that security. Thus, the underlying security and options written on that security can form the basis for the n-potential outcome economy. Consider a stock that will pay either 20, 40, or 60. Two calls are written on that stock, one with an exercise price of 30 and a second with an exercise price of 50:

$$\begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 10 \\ 30 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix},$$

$$\text{Stock} \quad X = 30 \quad X + 50.$$

These three securities form the basis for the three-dimensional state space. Any other security with a defined payoff vector in this three-potential outcome economy can be valued as a linear combination of these three securities. For example, consider a call option with an exercise price of 40. This option with a payoff vector of [0, 0, 20]′ can be replicated with a portfolio of payoff vectors forming the basis as follows:

$$\begin{bmatrix} 20 & 0 & 0 \\ 40 & 10 & 0 \\ 60 & 30 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} \#S \\ \#c_{X=30} \\ \#c_{X=50} \end{bmatrix}.$$

Thus, this call can be priced as a linear combination of the prices of the three securities forming the basis. One should always be able to form a basis for an n-state economy with a stock and $n - 1$ priced options written on that stock. Any other security (usually other options on that stock) whose payoff vectors can be defined for that economy can be priced as a linear combination of the prices of the securities forming the basis of payoff vectors for that economy.

EXERCISES

7.1. Add the following matrices:

(a) $\begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} =$; (b) $\begin{bmatrix} 4 & 6 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -6 \\ 5 & 0.5 \end{bmatrix} =$.

7.2. Transpose the following matrices:

(a) $\begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$; (b) $\begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$.

7.3. Sampson Company stock is currently selling for \$50 per share. One-year put and call options are selling on this stock with exercise prices equal to \$40 per share. Sampson Company stock may increase or decrease by either 20% or 40% per share over the next year. Thus, there are four possible outcomes for this stock. The riskless return rate is 10%. The call is currently selling for \$12.

(a) Write out the payoff vectors for the put, stock, riskless asset (face value of \$40) and the call.
(b) What is the current value of the put?

7.4. Multiply the following matrices:

(a) $\begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = ;$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = ;$

(c) $\begin{bmatrix} 0.04 & 0.04 \\ 0.04 & 0.16 \end{bmatrix}\begin{bmatrix} 33.3333 & -8.3333 \\ -8.3333 & 8.3333 \end{bmatrix} = ;$

(d) $[0.02 \quad 0.16 \quad 0.10]\begin{bmatrix} 0.02 \\ 0.16 \\ 0.10 \end{bmatrix} = ;$ (e) $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}[4 \quad 5 \quad 6] = .$

7.5. An investor has invested into three funds, fund A, fund B, and fund C. Each of these funds is comprised of three stocks, stock 1, stock 2, and stock 3. The portfolio weights for each of the stocks in each of the funds and the stock returns are given in the following tables:

	w_1	w_2	w_3
Fund A	0.15	0.25	0.60
Fund B	0.40	0.30	0.30
Fund C	0.30	0.25	0.45

Stock	Return
1	0.12
2	0.18
3	0.24

(a) Construct a single matrix of portfolio weights for the funds. Fund A will be represented in the first row, fund B in the second row, and fund C will be represented in the third row.
(b) Construct a column vector of stock returns.
(c) Multiply the weights matrix by the returns vector to obtain a column vector for returns on the two funds.
(d) Using matrix notation, demonstrate how one would find the return on the investor's overall portfolio if it were equally invested in the three funds.

7.6. The expected returns for three stocks in portfolio P are 0.07 for stock X, 0.09 for stock Y, and 0.13 for stock Z. The variance of returns for stock X

QRMC07 9/17/01 4:45 PM Page 139

is 0.04, 0.16 for stock Y, and 0.36 for stock Z. The covariance between returns on stocks X and Y is 0.01, 0.02 between stocks X and Z, and 0.08 between stocks Y and Z. Stock X comprises 30% of the portfolio, Y comprises 50% of the portfolio, and Z comprises 20%.

(a) Write an expected returns vector for the three stocks.
(b) Write the covariance matrix for the three stocks.
(c) Write the weights vector for the portfolio.
(d) What are the dimensions for the three matrices written for parts (a), (b), and (c)?
(e) Find the expected return of the portfolio using matrices written for parts (a), (b), and (c).
(f) Find the variance of returns for the portfolio using matrices written for parts (a), (b), and (c).

7.7. Invert the following matrices:

(a) $[8]$; (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$; (d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$;

(e) $\begin{bmatrix} 0.02 & 0.04 \\ 0.06 & 0.08 \end{bmatrix}$; (f) $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$;

(g) $\begin{bmatrix} 33.\overline{33} & -8.\overline{33} \\ -8.\overline{33} & 8.\overline{33} \end{bmatrix}$; (h) $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 20 \end{bmatrix}$.

7.8. Solve the following for **x**:

$$\underset{\mathbf{C}}{\begin{bmatrix} 33.\overline{3} & -8.\overline{3} \\ -8.\overline{3} & 8.\overline{3} \end{bmatrix}} \cdot \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\mathbf{s}.}{\begin{bmatrix} 0.01 \\ 0.11 \end{bmatrix}},$$

7.9. Solve the following for **x**:

$$\underset{\mathbf{C}}{\begin{bmatrix} 0.08 & 0.08 & 0.1 & 1 \\ 0.08 & 0.32 & 0.2 & 1 \\ 0.1 & 0.2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}} \cdot \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{\mathbf{s}.}{\begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}},$$

7.10. The Victoria Company's financial statements are given below. Management is forecasting an increase in the company's sales level by 50% to \$1,125,000. Managers predict that this 50% sales increase will increase

QRMC07 9/17/01 4:45 PM Page 140

the firm's Cost of Goods Sold level by 50% to $450,000. Fixed costs will remain constant at $150,000. The firm will continue to make the $50,000 interest payments necessary to sustain its current $600,000 in bonds outstanding. Management expects the firm to remain in the 40% corporate income tax bracket and pay out one third of its earnings in dividends. In order to sustain this 50% increase in sales, management has determined that each asset account must also increase by 50%; that is, the total must increase by $500,000. Current Liabilities will also increase by 50%. The firm pays no interest on its Current Liabilities. Managers have already decided to sell bonds at an interest rate of 10% to provide any external capital necessary to finance the asset level increase. Management's problem is to determine how much additional capital to raise through this 10% bond issue. Based on this information and the company's financial statements given below, determine the Victoria Company's 2001 external funding needs (*EFN*).

VICTORIA COMPANY FINANCIAL STATEMENTS

Income statement, 2000		*Pro-forma income statement, 2001*	
Sales (*TR*)	$750,000	Sales (*TR*)	$1,125,000
Cost of Goods Sold	300,000	Cost of Goods Sold	450,000
Gross Margin	450,000	Gross Margin	675,000
Fixed Costs	150,000	Fixed Costs	150,000
EBIT	300,000	*EBIT*	525,000
Interest Payments	50,000	Interest Payments	
Earnings Before Taxes	250,000	Earnings Before Taxes	
Taxes (@ 40%)	100,000	Taxes (@ 40%)	
Net Income After Tax	150,000	Net Income After Taxes	
Dividends (@ 33%)	50,000	Dividends (@ 33%)	
Retained Earnings	100,000	Retained Earnings	

Balance sheet, December 31, 2000

ASSETS		LIABILITIES AND EQUITY	
Cash	$100,000	Accounts Payable	$100,000
Accounts Receivable	100,000	Accrued Wages	50,000
Inventory	100,000	Current Liabilities	150,000
Current Assets	300,000	Bonds Payable	600,000
Plant and Equipment	700,000	Equity	250,000
Total Assets	1,000,000	Total Capital	1,000,000

Pro-forma balance sheet, December 31, 2001

ASSETS		LIABILITIES AND EQUITY	
Cash	$150,000	Accounts Payable	$150,000
Accounts Receivable	150,000	Accrued Wages	75,000
Inventory	150,000	Current Liabilities	225,000
Current Assets	450,000	Bonds Payable	
Plant and Equipment	1,050,000	Equity	
Total Assets	1,500,000	Total Capital	1,500,000

7.11. The following table reflects riskless bond prices, coupon rates and terms to maturity for a given economy:

Bond	Price	Coupon rate (%)	Years to maturity
A	1,000	10	1
B	980	10	2
C	960	10	3
D	940	10	4

Assuming the bonds make annual coupon payments at year end, answer the following questions based on the above information:

(a) What are spot rates for years one through four?
(b) What is the two-year forward rate for a loan originated in one year?

7.12. Consider two three-year bonds with \$1,000 face values. The coupon rate of bond X is 5% and 8% for bond Y. Now consider a third bond Z with coupon rate of 11% and a maturity of two years. Bond Z's face value is also \$1,000. The current market prices of bonds X, Y, and Z are \$878.9172, \$955.4787, and \$1,055.419, respectively.

(a) What are the spot rates implied by these bonds?
(b) Find a portfolio of bonds X, Y, and Z which would replicate the cash flow structure of bond Q, which has a face value of \$1,000, a maturity of three years, and a coupon rate of 15%.

7.13. A pension fund expects to make payments of \$80,000,000 in one year, \$100,000,000 in two years, \$120,000,000 in three years, and \$140,000,000 in four years to shareholders in the fund. These anticipated cash flows are to be matched with a portfolio of the following \$1,000 face value bonds:

Bond	Current price	Coupon rate	Years to maturity
1	1,000	0.10	1
2	980	0.10	2
3	1,000	0.11	3
4	1,000	0.12	4

How many of each of the four bonds should the fund purchase to exactly match its anticipated payments to shareholders?

7.14. Bond A, a two-year, 12% coupon issue can be purchased for \$957.9920. Bond B, a two-year, 5% coupon issue can be purchased for \$840.2471.

QRMC07 9/17/01 4:45 PM Page 142

(a) What is the one-year spot rate of interest ($y_{0,1}$)?
(b) What is the two-year spot rate of interest ($y_{0,2}$)?
(c) What would be the value of a $1,000 face value pure discount bond maturing in two years?
(d) The two-year pure discount bond in part (c) can be replicated with a portfolio comprised of bonds A and B. What should the portfolio weights of these bonds be? (Short selling is permitted.)
(e) I need to raise $15,000 at the end of year one and $12,000 at the end of year two to repay some debts. How much should I buy (sell) of each of bonds A and B to exactly match my debt payments? (Fractional purchases and sales of bonds are permitted.)

7.15. Buford Company stock currently sells for $24 per share and is expected to be worth either $20 or $32 in one year. The current riskless return rate is 0.125. What would be the value of a one-year call with an exercise price of $16?

7.16. Robinson Company stock currently sells for $20 per share and will pay off either $15 or $25 in one year. A one-year call with an exercise price equal to $18 has been written on this stock. This call sells for $3.5.

(a) What is the value of a one-year call with an exercise price equal to $22?
(b) What is the riskless return rate for this economy?
(c) What is the value of a one-year put that can be exercised for $40?

APPENDIX 7.A MATRIX MATHEMATICS ON A SPREADSHEET

(Background reading: sections 7.3, 7.4, and 7.5 and appendix 3.A)

Solving equations with matrices of higher order by hand or with a calculator can be an extremely time-consuming and frustrating process. However, spreadsheets can be used quite effectively to multiply and invert matrices. Suppose that we wished to solve the following for **x** using an Excel™ spreadsheet:

$$\begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix},$$

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{s}.$$

We may start to solve this system by insert the coefficients from matrix **C** in the spreadsheet as follows:

QRMC07 9/17/01 4:45 PM Page 143

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4					
5					

To solve the system, we will first invert matrix **C** in cells A1:B2. First, use the mouse to highlight cells A4 to B5, where we will insert the inverse of matrix **C**. We then select from the toolbar at the top of the screen the Paste Function button ($\boldsymbol{f_x}$). From the Paste Function menu, we select the MATH & TRIG sub-menu. In the MATH & TRIG sub-menu, we scroll down to select MINVERSE, the function, which inverts the matrix. The MINVERSE function will prompt for an array; we enter the location of the matrix to be inverted: A1:B2. To fill all four cells A4 to B5, we simultaneously hit the Ctrl, Shift, and Enter keys. This is important. The spreadsheet should then appear as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4	0.15	–0.1			
5	–0.05	0.2			

The matrix in cells A4:B5 is $\mathbf{C}^{-1}$. Now, we enter into cells C4 and C5 the equation solution vector **s** containing elements 10 and 20. Then highlight cells D4 and D5 to solve for vector **x**, left click again the Paste Function key in the Toolbar and select the MATH & TRIG menu. Then scroll down to and select the MMULT function, which will enable us to premultiply our solutions vector **s** by matrix $\mathbf{C}^{-1}$. The dialogue box will prompt for two arrays. The first will be matrix $\mathbf{C}^{-1}$ in cells A4:B5. Then hit the Tab key and enter the cells for the second array C4:C5. Then hit the Ctrl, Shift, and Enter keys simultaneously to fill cells D4 and D5. The result will be vector **x** with elements −0.5 and 3.5. Thus, $x_1 = -0.5$ and $x_2 = 3.5$. The final spreadsheet will appear as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4	0.15	–0.1	10	–0.5	
5	–0.05	0.2	20	3.5	

The process of expanding this solution procedure to larger matrices is quite simple. First, be certain that each equation in the system is linear (no exponents other than 0 or 1 on the variables) and that the coefficient matrix is square. In many cases, the systems cannot be solved. Among these are the following: the coefficients matrix is not

QRMC07 9/17/01 4:45 PM Page 144

square; matrices do not conform for multiplication; or the coefficients matrix is singular. Consider the following fourth order system:

$$\begin{bmatrix} 8 & 4 & 2 & 10 \\ 2 & 4 & 1 & 12 \\ 0 & 4 & 2 & 16 \\ 5 & 6 & 8 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix},$$

$$\mathbf{C} \cdot \mathbf{x} = \mathbf{s}.$$

Now, examine the following spreadsheet, which is used to solve the system:

	A	B	C	D	E	F	G
1	8	4	2	10			
2	2	4	1	12			
3	0	4	2	16			
4	5	6	8	20			
5							
6	0.235294	−0.29412	0.147059	−0.05882	10	−1.470588	
7	−0.52941	1.578431	−1.12255	0.215686	20	1.2254902	
8	−0.11765	−0.01961	−0.15686	0.196078	30	1.5686275	
9	0.147059	−0.39216	0.362745	−0.07843	40	1.372549	

Thus, $x_1 = -1.470588$, $x_2 = 1.2254902$, $x_3 = 1.5686275$, and $x_4 = 1.372549$.