Chapter 10: The Black-Scholes Model

A. Preliminaries

In Chapters 2, 3, 7 and 8 of this manuscript, we discussed several models for pricing options. The pure security methodologies in Chapters 2 and 3 are particularly instructive because they highlight the importance of no-arbitrage in option pricing. That is, physical probabilities are not needed when capital markets are complete and when the Equivalent Martingale can be applied. We continued this discussion into Chapter 8, where we discussed option pricing in binomial environments. The binomial model can be extended into a Black-Scholes continuous time environment with application of the Central Limit Theorem. In Chapter 7, paying only cursory attention to arbitrage and the importance of risk-neutral pricing, we heuristically derived the Black-Scholes model. In Chapter 9, we developed the mathematical methodology to rigorously derive from arbitrage-free perspectives the Black-Scholes model and many other models. We will apply this more rigorous methodology in this chapter.

Here, we will employ the analytical approach of Black and Scholes themselves, where the option value is the solution to the appropriate boundary value problem. We will discuss estimating unobservable underlying security volatility, an essential input of the model, the model's sensitivities to its inputs and then a variety of applications and extensions of the model.

<u>Self-Financing Strategies and Portfolios</u>

In this section, we are going to reintroduce the self-financing replicating portfolio and how to use it to price derivatives. We will assume a market consisting of a stock and a riskless bond that will be used to create the portfolio. For convenience, we will illustrate the concepts using a European call as our derivative, but the method will be the same for many types of derivatives. Let c_t denote the price of the call at time *t.* Consider a portfolio ($\gamma_{s,t,}$, $\gamma_{b,t}$) combining $\gamma_{s,t}$ shares of stock at a per share price S_t at time *t* and $\gamma_{b,t}$ units of the riskless bond with a price per unit of B_t at time *t*. The value of the portfolio at time *t* is then: $V_t = \gamma_{s,t,}S_t+\gamma_{b,t}B_t$. Assume thatwe purchase the call at time 0 and *T* is the call expiry and bond maturity date. On the expiry date, the value of the European call will be known: $c_T = max(0,S_T - X)$, where *X* is the exercise price of the call.

We say that the portfolio ($\gamma_{s,t,}$, $\gamma_{b,t}$) is a *self-financing replicating portfolio* for the call if and only if the following two properties are satisfied:

I $$dV_t = \gamma_{s,t}dS_t + \gamma_{b,t}dB_t,$$

and

II $$c_T = \gamma_{s,T}S_T + \gamma_{b,T}B_T.$$

Property I is called the self-financing property. The interpretation of Property I is that during every infinitesimal time interval (*t,t+dt*) the change in the value of the portfolio is entirely due to the changes in the prices of the stock and bond. There is no net new investment in the portfolio. In other words, any infinitesimal purchases or sales of the stock and bond ($d\gamma_{s,t}$ and $d\gamma_{b,t}$) will offset each other so that the change in the value of the portfolio is entirely due to the changes in the value of the securities themselves. Property II states that the expiry date T value of the replicating portfolio will equal the price of the call at expiry date *T*. We will also assume the absence of arbitrage opportunities.

We will create a self-financing portfolio, consisting of a single T-period call, the underlying stock, and a T-period riskless bond. We will see that this arbitrage portfolio will have zero net investment from time 0 to time T, and it will be shown that its value is always equal to 0. This portfolio $(-1, \gamma_{s,t}, \gamma_{b,t})$ will consist of a short position in a single call along with positions in $\gamma_{s,t}$ shares of underlying stock and $\gamma_{b,t}$ units of the riskless bond. We shall see soon that our short position in the call will be offset by a long position in the underlying stock ($\gamma_{s,t}$ will be positive), and a short position in $\gamma_{b,t}$ units of the riskless bond ($\gamma_{b,t}$ will be negative). Denote the values at time t of each unit of the call, stock, and bond by c_t (whose value is not yet known), S_t, and B_t, respectively. If P_t is the value of the portfolio, then

(1) $$P_t = -c_t + \gamma_{s,t} S_t + \gamma_{b,t} B_t.$$

Assume that we purchase the call at time 0 and T is the expiry date. On the expiry date, the value of the call will be known and the bond will mature. For example, a European call will be worth $c_T = max(0,S_T - X)$, where X is the exercise price of the call. For our portfolio, the number of shares $\gamma_{s,T}$ of stock and the number of units of the bond $\gamma_{b,T}$ will be chosen so that the portfolio's expiry date value will be P_T:

(2) $$P_T = -c_T + \gamma_{s,T} S_T + \gamma_{b,T} B_T = 0.$$

We will also determine the values of $\gamma_{s,t}$ and $\gamma_{b,t}$ at every moment t so that during every infinitesimal time interval (t, $t+dt$) there is zero net new investment in the portfolio. In other words, any infinitesimal purchases or sales of the stock and bond ($d\gamma_{s,t}$ and $d\gamma_{b,t}$) will offset each other so that the change in the value of the portfolio is entirely due to the changes in the value of the securities themselves. This can be expressed mathematically as:

(3) $$dP_t = -dc_t + \gamma_{s,t} dS_t + \gamma_{b,t} dB_t$$

for any time $t, 0 \le t \le T$. In a no-arbitrage market, conditions suggested by equations (2) and (3) guarantee that the owner of the portfolio requires no capital at all to construct and maintain the portfolio. Such a portfolio is called a self-financing portfolio.[1] In an arbitrage-free market, this implies that the value of the portfolio P_t equals zero for all time $t,\ 0 \le t \le T$. The reason is quite simple. Suppose at some point in time t the value of the portfolio was negative: $P_t = -c_t + \gamma_{s,t}S_t + \gamma_{b,t}B_t < 0$. By equation (2), a long position in this portfolio would certainly produce a riskless profit by time T, since the portfolio is constructed to have zero value at time T. Since the portfolio's current price is negative, its purchase would produce a positive time t cash flow. Since the portfolio satisfies the condition given by equation (3), we would not need to use any capital to maintain the portfolio from time t until the option expiry date T. Assuming an interest rate r, a guaranteed profit of $(\gamma_{s,t}S_t+\gamma_{b,t}B_t+c_t)e^{r(T-t)}$ with interest is locked in by the expiry date with no positive net investment. Of course, this would violate our no-arbitrage principle. Similarly, if the value of the portfolio were positive at any time t, we would short the portfolio to produce a time T arbitrage profit by simply reversing the positions taken by buying the portfolio. Once again, this would result in a guaranteed profit with no net expenditure on our part, violating the no-arbitrage requirement. Thus, we are able to conclude that

[1] Many authors only require condition (3) for the definition of a self-financing portfolio.

(4) $$P_t = -c_t + \gamma_{s,t}S_t + \gamma_{b,t}B_t = 0$$

for all time *t, 0* ≤ *t* ≤ *T.* We can solve this equation for the price of the call at time *t*:

(5) $$c_t = \gamma_{s,t}S_t + \gamma_{b,t}B_t.$$

There is still a lot of work to do to obtain a numerical solution for the value of the call, but we have reduced the problem to ensuring that the call must satisfy equations 2 and 3.

The arbitrage-free portfolio in the previous section always meets the two requirements: $-c_T + \gamma_{s,T}S_T + \gamma_{b,T}B_T = 0$ and $-dc_t + \gamma_{s,t}dS_t + \gamma_{b,t}dB_t = 0$ for $0 \le t \le T$. Since the derivation is the same for any derivative for which such hedging portfolios exist, we will write the conditions more generally for any derivative whose value at time *t* will be denoted by V_t:

(6) $$V_t = \gamma_{s,t}S_t + \gamma_{b,t}B_t$$

and

(7) $$dV_t = \gamma_{s,t}dS_t + \gamma_{b,t}dB_t.$$

for $0 \le t \le T$. From this perspective, we are viewing the portfolio ($\gamma_{s,t,}$, $\gamma_{b,t}$) of stock and bonds as replicating the derivative. Equation (7) portrays the self-financing property of the replicating portfolio. In an arbitrage-free market, we showed in the previous section that equations (6) and (7) imply that the derivative value equals the value of the portfolio for all time *t* (equation 5). For this reason, a portfolio ($\gamma_{s,t,}$, $\gamma_{b,t}$) that satisfies equations (6) and (7) is called a *self-financing replicating portfolio* for the derivative with value V_t. We have shown that if we can create a self-financing replicating portfolio for the derivative that we know how to price, then it must equal the arbitrage free price for the derivative. We are going to use two different approaches to derive the price of the derivative. The first approach will use martingales and the second approach will derive a partial differential equation known as the Black-Scholes equation that the derivative must satisfy.

Pricing a European Call Option and the Black-Scholes Formula

Now, we will be specific in our choice of derivatives; we will value a European call. First, we specify its payoff function at time *T* when it might be exercised. With exercise price *X*, the value of the option at expiry is $c_T = MAX[S_T - X, 0]$. The value in risk neutral probability space of the call at time 0 is

$$c_0 = e^{-rT}E_{\mathbb{Q}}[V_T|\mathcal{F}_0] = e^{-rT}E_{\mathbb{Q}}[c_T|\mathcal{F}_0] = e^{-rT}E_{\mathbb{Q}}[\text{MAX}(S_T - X, 0)].$$

We showed earlier that

$$d(e^{-rt}S_t) = \sigma e^{-rt}S_t d\hat{Z}_t.$$

By the special product rule for stochastic differentials in Section A of Chapter 9, we have

$$e^{-rt}dS_t - re^{-rt}S_t dt = \sigma e^{-rt}S_t d\hat{Z}_t.$$

Solving for dS_t gives:

$$dS_t = rS_t dt + \sigma S_t d\hat{Z}_t.$$

As we showed in Section C of Chapter 9 leading to Equation 15), with the choice of $\mu = r$, the solution is:

$$S_T = S_0 e^{\sigma \hat{Z}_T + \left(r - \frac{1}{2}\sigma^2\right)T}.$$

Recall that we used a slight variation of this equation in Chapter 9 to value a call. Since $\hat{Z}_T \sim N(0,T)$, then $\hat{Z}_T = Z\sqrt{T}$ with $Z \sim N(0,1)$. Thus, the solutions S_T here and in Chapter 7 have identical probability distributions. Since the value of the call at time 0 is the same expected value: $c_0 = E_{\mathbb{Q}}[c_T] = E_{\mathbb{Q}}[MAX(S_T - X, 0)]$,this leads to the Black-Scholes formula, the same result that we obtained in Chapter 7, expressed as an expected future value:

$$S_0 e^{rT} N(d_1) - XN(d_2).$$

In Chapters 7 and 9, and earlier in this chapter, we have set forth the mathematics and pricing framework to value options and other derivatives. Our derivation of the Black-Scholes model in Chapter 7, while not incorrect, failed to show why we needed to change our probability measure to risk-neutral space and focus on the riskless return rather than the expected risk-adjusted return of the underlying stock. Now, it should be clear that our analysis and derivation needs to focus on the expectation $E_{\mathbb{Q}}[MAX(S_T - X, 0)]$, even though its computation is exactly as we demonstrated in Chapter 7. The key to this and the previous chapter is that we have demonstrated the critically important feature of this expectation in risk-neutral space: this expectation provides an arbitrage-free pricing of the call option or other derivative for which we can find a self-financing replicating portfolio. Since the portfolio is self-financing, it requires zero net investment to maintain the arbitrage position, and since it is replicating, its initial value must be the same as the call's.

B. Deriving the Black-Scholes Model

In this section, we discuss the Black-Scholes derivation of their option pricing model. We then work through some very simple illustrations.

<u>Black-Scholes Assumptions</u>

Black and Scholes [1973] set forth a rather strict set of assumptions for their model (the same assumptions apply to the martingale derivations). Most importantly, the model assumes that underlying share prices follow a geometric Brownian motion process and that investors can create hedged self-financing portfolios comprising calls, underlying shares and riskless bonds. The set of assumptions on which the Black-Scholes model and its derivation are based are as follows:

1. There exist no restrictions on short sales of stock or writing of call options.
2. There are no transactions costs.

3. There exists continuous trading of stocks and options.
4. There exists a known constant riskless borrowing and lending rate r.
5. The underlying stock will pay no dividends or make other distributions during the life of the option.
6. The option can be exercised only on its expiration date; that is, it is a European Option.
7. Shares of stock and option contracts are infinitely divisible.
8. Underlying stock prices follow a geometric Brownian motion process: $dS_t = \mu S_t dt + \sigma S_t dZ_t$. with constant μ and σ extending over the life of the option.

It is important to note that, because of the Cox-Ross Risk Neutrality argument (change of measure to risk-neutral space), the following are not required as model inputs:

1. The expected or required return or risk premium on the stock or option and
2. Investor attitudes toward risk

The Self-Financing Replicating Portfolio and Black-Scholes

In the previous section, we priced a derivative using a self-financing replicating portfolio and martingales. Again in this section, we will price a derivative instrument using a self-financing replicating portfolio. However, rather than follow the methodologies in Chapters 7 and 8, we will derive and solve the appropriate partial differential equation (known as the Black-Scholes differential equation) that the derivative instrument must satisfy. This partial differential equation can then be solved by standard techniques (as set forth in the appendix to this chapter) to obtain its price. In particular, we will derive the Black-Scholes option pricing model for a call, assuming that our all standard Black-Scholes assumptions hold. As we learned in Section A, to price a call c_t at any time $t < T$, it is sufficient to construct a self-financing replicating portfolio $(\gamma_{s,t},\gamma_{b,t})$ of stocks and bonds whose value equals the value of the call at time T. If we denote the value of the portfolio by

(14) $$c_t = \gamma_{s,t} S_t + \gamma_{b,t} B_{t,}$$

with c_T equal to the value of the call at expiry, and we require that the self-financing property is satisfied:

(15) $$dc_t = \gamma_{s,t} dS_t + \gamma_{b,t} dB_t.$$

It is convenient to rewrite the self-financing property (15) in the form:

(16) $$\gamma_{b,t} dB_t = dc_t - \gamma_{s,t} dS_t.$$

Since $B_t = e^{rt}$, $dB_t = re^{rt}dt = rB_t\,dt$, so equation (16) takes the form

(17) $$r\gamma_{b,t} B_t dt = dc_t - \gamma_{s,t} dS_t.$$

Solving for $\gamma_{b,t}B_t = c_t - \gamma_{s,t}S_t$ in equation (14), and substituting this result into equation (17), equation (17) becomes:

(18) $$r(c_t - \gamma_{s,t}S_t)dt = dc_t - \gamma_{s,t}dS_t.$$

Recall that we assume that the stock price follows a geometric Brownian motion process:

$$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

Now, we will invoke Itô's Lemma from Chapter 9 to express the differential dc_t (equation 15):

(19) $$dc_t = \left(\frac{\partial c}{\partial t} + \mu S_t \frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2}\right) dt + \sigma S_t \frac{\partial c}{\partial S} dZ_t,$$

which we will use to replace dc_t in equation (18) and then simplify:

(20) $$\begin{aligned} r(c_t - \gamma_{s,t}S_t)dt &= \left(\frac{\partial c}{\partial t} + \mu S_t \frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2}\right) dt + \sigma S_t \frac{\partial c}{\partial S} dZ_t - \gamma_{s,t}dS_t \\ &= \left(\frac{\partial c}{\partial t} + \mu S_t \frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2}\right) dt + \sigma S_t \frac{\partial c}{\partial S} dZ_t - \gamma_{s,t}(\mu S_t dt + \sigma S_t dZ_t) \\ &= \left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2}\right) dt + \mu S_t \left(\frac{\partial c}{\partial S} - \gamma_{s,t}\right) dt + \sigma S_t \left(\frac{\partial c}{\partial S} - \gamma_{s,t}\right) dZ_t. \end{aligned}$$

The instantaneous expected rate of return μ may reflect individual investor forecasts and risk preferences, and might even vary from investor to investor. It is unobservable and not very useful for most option valuation calculations. We encountered this situation before with physical probabilities. We dealt with this issue earlier by using Radon-Nikodym derivatives (we used pure securities and hedging probabilities without technically using the term Radon-Nikodym derivative) to transform physical probabilities into risk-neutral probabilities, which are consistent with eliminating arbitrage opportunities. If we choose $\gamma_{s,t} = \frac{\partial c}{\partial S}$, equation (20) will simplify to:

(21) $$r\left(c_t - S_t \frac{\partial c}{\partial S}\right) dt = \left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2}\right) dt.$$

This step is key. Notice that this Equation (21) makes no reference to μ, the instantaneous expected rate of return for the stock. This suggests our pricing model will not depend on a risk premium or investor risk preferences, as in fact we will see when we obtain the solution. Only the riskless return r is used. One can view the amount $c_t - S_t \frac{\partial c}{\partial S}$ as the value of a portfolio consisting of buying 1 call and shorting $\frac{\partial c}{\partial S}$ shares of the stock. As we stated above, this portfolio is known as a delta hedged portfolio. It is perfectly hedged to guarantee the riskless rate of return r. Equation (21) is divided by dt to become:

$$r\left(c_t - S_t \frac{\partial c}{\partial S}\right) = \frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2}.$$

If the stock price is known (e.g., if $S = S_0$, today's stock price), the call value becomes a function of the real-valued independent variables S and t:

(22) $$r\left(c - S\frac{\partial c}{\partial S}\right) = \frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}.$$

or

(23) $$\frac{\partial c}{\partial t} = rc - rS\frac{\partial c}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2}.$$

This partial differential equation is called the *Black-Scholes differential equation*. To obtain a unique solution to the Black-Scholes equation, we need appropriate boundary conditions. Since there is a first-order partial derivative of *c* with respect to the variable *t*, we need one boundary condition for *c* with respect to time. In our case, this will be to specify the value at the time of expiration *T*. Of course, the value of the call $c(S,T)$ at expiry time *T* is a known function of the stock price *S:* $c(S,T) = max(S\text{-}X,0)$. Since there is a second-order partial derivative of *c* with respect to *S* (it is the highest order derivative that matters), then we need to specify two boundary conditions for *c* with respect to the stock price. Obviously, if the stock price $S = 0$, the call should be worthless at any time. So we will assume the boundary condition $c(0,t) = 0$. As $S\rightarrow\infty$, the stock price will always be able to overtake any exercise price *X* on the expiration date. This leads to the boundary condition $c(S,t)\rightarrow S\text{-}X$ as $S\rightarrow\infty$. The Black-Scholes equation and these boundary conditions guarantee that there is a unique solution (unique pricing) for the call. For the interested reader, a derivation of the solution is in appendix A to this chapter.

The Black Scholes Model

Consistent with our findings in Chapters 7 and 9, the value of the call at time zero is:

(24) $$c_0 = S_0 N(d_1) - Xe^{-rT}N(d_2)$$

with

(25) $$d_1 = d_2 + \sigma\sqrt{T} \text{ and } d_2 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

where $N(d^*)$ is the cumulative normal density function for (d^*). From a computational perspective, one would first solve for d_1 and d_2 before c_0.

Again, it is important to note how far-ranging the implications are for the Black-Scholes differential equation. While we have focused on call valuation thus far, the equation can be used to value any derivative instrument on the underlying security whose price is generated by $dS_t = \mu S_t dt + \sigma S_t dZ_t$. We will discuss additional applications of the Black-Scholes differential equation to additional derivative instruments later in this chapter.

Put-Call Parity

From initial payoff functions set forth in Chapter 6, the terminal value *put-call relation* is characterized as follows:

(6) $$c_T - p_T = MAX[0, S_T - X] - MAX[0, X - S_T] = S_T - X$$

A slight rewrite of this terminal put-call relation allows us to write the terminal or exercise value a put given the terminal value of a call with identical exercise terms:

(4) $$p_T = c_T + X - S_T$$

The Black-Scholes Model: Simple Numerical Illustrations

Consider the following example of a Black-Scholes model application where an investor can purchase a six-month call option for \$7.00 on a stock that is currently selling for \$75. The exercise price of the call is \$80 and the current riskless rate of return is 10% per annum. The variance of annual returns on the underlying stock is 16%. At its current price of \$7.00, does this option represent a good investment? We will note the model inputs in symbolic form:

$T = .5$	$r = .10$
$X = 80$	$\sigma^2 = .16$
$\sigma = .4$	$S_0 = 75$

Our first step is to find d_1 and d_2:

$$d_1 = \frac{ln\left(\frac{75}{80}\right) + \left(.10 + \frac{1}{2} \times .16\right) \times .5}{.4\sqrt{.5}} = \frac{ln(.9375) + .09}{.2828} = .09$$

$$d_2 = .09 - .4\sqrt{.5} = .09 - .2828 = -.1928$$

Next, by either using a *z*-table (See end-of-chapter Appendix B) or by using an appropriate polynomial estimating function, we find cumulative normal density functions for d_1 and d_2:

$$N(d_1) = N(.09) = .535864$$
$$N(d_2) = N(-.1928) = .423549$$

Finally, we use $N(d_1)$ and $N(d_2)$ to value the call:

$$c_0 = 75 \times .536 - \frac{80}{e^{.10 \times .5}} \times .424 = 7.958$$

Since the 7.958 value of the call exceeds its 7.00 market price, the call represents a good purchase. Next, we use the put-call parity relation to find the value of the put as follows:

$$p_0 = c_0 + Xe^{-rT} - S_0$$

$$p_0 = 7.958 + 80(.9512) - 75 = 9.054$$

The next Section C will focus on issues related to the application of Black-Scholes to option pricing and to variance estimates. In later chapters, we will begin to relax Black-Scholes assumptions to obtain additional applications of Black-Scholes.

C. Implied Volatility

Four of the 5 inputs required to implement the Black-Scholes model are easily observed. The option exercise price and term to expiry are defined by the option contract. The riskless return and underlying stock price are based on current quotes. Only the underlying stock return volatility during the life of the option cannot be observed. Instead, we often employ a traditional sample estimating procedure for return variance:

$$\sigma^2 = Var[r_t] = Var[lnS_t - lnS_{t-1}]$$

The difficulty with this procedure is that it requires that we assume that underlying security return variance is stable over time; more specifically, that future variances equal or can be estimated from historical variances. An alternative procedure suggested by Latane and Rendleman [1976] is based on market prices of options that might be used to imply variance estimates. For example, the Black-Scholes Option Pricing Model might provide an excellent means to estimate underlying stock variances if the market prices of one or more relevant calls and puts are known. Essentially, this procedure determines market estimates for underlying stock variance based on known market prices for options on the underlying securities. When we use this procedure, we assume that the market reveals its estimate of volatility through the market prices of options.

Consider the following example pertaining to a six-month call currently trading for $8.20 and its underlying stock currently trading for $75:

$T = .5$	$r = .10$	$c_0 = 8.20$
$X = 80$	$S_0 = 75$	

If investors use the Black-Scholes Options Pricing Model to value calls, the following should be expected:

$$8.20 = 75N(d_1) - 80e^{-.1\times.5}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{75}{80}\right) + \left(.1 + .5\times\sigma^2\right)\cdot .5}{\sigma\sqrt{.5}}$$

$$d_2 = d_1 - \sigma\sqrt{.5}$$

As we will demonstrate shortly, we find that this system of equations holds when σ = .41147. Thus, the market prices this call as though it expects that the standard deviation of anticipated returns for the underlying stock is .41147.

Unfortunately, the system of equations required to obtain an implied variance has no closed form solution. That is, we will be unable to solve this equation set explicitly for standard deviation; we must search, iterate and substitute for a solution. One can substitute trial values for σ until she finds one that solves the system. A significant amount of time can be saved by using one of several well-known numerical search procedures such as the Method of Bisection or the Newton-Raphson Method.

The Method of Bisection

We seek to solve the above system of equations for σ. This is equivalent to solving for the root of:

$$f(\sigma^*) = 0 = 75\times N(d_1) - 80\times e^{-.1\times.5}\times N(d_2) - 8.20$$

based on equations above for d_1 and d_2. There exists no closed form solution for σ. Thus, we will use the Method of Bisection to search for a solution. We first arbitrarily select endpoints for our range of guesses, such as b_1=.2 and a_1=.5 so that $f(b_1)$ = -4.46788 < 0 and $f(a_1)$ = 1.860465 > 0. Since these endpoints result in $f(\sigma)$ with opposite signs, our first iteration will be in the middle: σ_1 = .5(.2+.5) = .35. We find that this estimate for σ results in a value of -1.29619 for $f(\sigma)$. Since this $f(\sigma)$ is negative, we know that σ^* is in the segment b_2=.35 and a_2=.5. Moving to Row n=2, we repeat the iteration process, finding after 16 iterations that σ^*=.41146. Table 1 details the process of iteration.

Equation for f: $S_0N(d_1)-Xe^{-rt}N(d_2)-c_0$

a_1 = 0.5 b_1= 0.2 σ_1 = 0.35 r = 0.1 S_0= 75 X = 80
c_0= 8.2 T = 0.5

	σ_n	$d_1(\sigma_n)$	$d_2(\sigma_n)$	$N(d_1)$	$N(d_2)$	$N(d_1)$	$N(d_2)$	$f(\sigma_n)$
$f(a_1)$= 1.860465	0.5	0.1356555	-0.2178978	0.553953	0.41375	0.553953	0.413754	1.860465
$f(b_1)$= - 4.46788	0.2	-0.0320922	-0.1735135	0.487199	0.431122	0.487199	0.431124	-4.46788

n	a_n	b_n	σ_n	$d_1(\sigma_n)$	$d_2(\sigma_n)$	$N(d_1)$	$N(d_2)$	$N(d_1)$	$N(d_2)$	$f(\sigma_n)$
1	0.5	0.2	0.35	0.06499919	-0.1824882	0.525913	0.427597	0.525913	0.4276	-1.29619
2	0.5	0.35	0.425	0.10188237	-0.198638	0.540575	0.42127	0.540575	0.421273	0.284948
3	0.425	0.35	0.3875	0.08394239	-0.1900615	0.533449	0.424628	0.533449	0.424630	-0.50501
4	0.425	0.3875	0.40625	0.09302042	-0.1942417	0.537056	0.42299	0.537056	0.422993	-0.10987
5	0.425	0.40625	0.41562	0.09747658	-0.1964147	0.538826	0.42214	0.538826	0.422143	0.087583
6	0.41562	0.40625	0.41093	0.09525501	-0.1953217	0.537944	0.42256	0.537944	0.422571	-0.01113
7	0.41562	0.41093	0.41328	0.09636739	-0.1958666	0.538386	0.42235	0.538386	0.422357	0.038229
8	0.41328	0.41093	0.41210	0.09581161	-0.1955937	0.538165	0.42246	0.538165	0.422464	0.01355
9	0.41210	0.41093	0.41152	0.09553341	-0.1954576	0.538054	0.42251	0.538054	0.422517	0.00121
10	0.41152	0.41093	0.41123	0.09539424	-0.1953896	0.537999	0.42254	0.537999	0.422544	-0.00496

11	0.41152	0.41123	0.41137	0.09546383	-0.1954236	0.538027	0.42252	0.538027	0.422531	-0.00188
12	0.41152	0.41137	0.41145	0.09549862	-0.1954406	0.538041	0.42252	0.538041	0.422524	-0.00033
13	0.41152	0.41145	0.41148	0.09551602	-0.1954491	0.538048	0.42251	0.538048	0.422521	0.000438
14	0.41148	0.41145	0.41146	0.09550732	-0.1954449	0.538044	0.42251	0.538044	0.422522	0.000053
15	0.41146	0.41145	0.41145	0.09550297	-0.1954427	0.538042	0.42252	0.538042	0.422523	-0.00014
16	0.41146	0.41145	0.41146	0.09550514	-0.1954438	0.538043	0.42252	0.538043	0.422523	-0.00004

Table 1: Using the Bisection Method to Estimate Implied Volatility

The Newton Raphson Method

The Newton-Raphson Method can also be used to more efficiently iterate for an implied volatility. We will solve for the implied standard deviation in our illustration using the Newton-Raphson Method to find the root of the equation: $f(\sigma) = S_0N(d_1) - Xe^{-rT}N(d_2) - c_0$. The Newton Raphson Method estimates the root of an equation f(σ) =0 by using the formula

$$\sigma_n = \sigma_{n-1} - \frac{f(\sigma_{n-1})}{f'(\sigma_{n-1})}.$$

One starts with some initial rough estimate σ_0 for the root, then repeatedly applying the formula above, iterating until the desired accuracy is obtained.

For our example, we arbitrarily choose an initial trial solution of $\sigma = \sigma_0 = .6$. First, we need the derivative of $f(\sigma)$ with respect to the underlying stock return standard deviation σ.[2] Since we are treating c_0 as a given constant, differentiating this function is equivalent to differentiating the call function $c = S_0N(d_1) - Xe^{-rT}N(d_2)$ with respect to σ. We leave as a homework exercise that:[2]

$$\frac{\partial c}{\partial \sigma} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} > 0, \qquad \textit{Vega } \nu$$

with an arbitrarily selected initial trial solution of $\sigma_0 = .6$. We see from Table 2 that this standard deviation results in a value of $f(\sigma_0) = 3.95012$, implying a variance estimate that is too high. Substituting .6 into Equation 3 for σ_0, we find that $f'(\sigma_0) = 20.82509$. Thus, our second trial value for σ is determined by: $\sigma_1 \approx \sigma_0 - (f(\sigma_0) \div f'(\sigma_0)) = .6 - (3.95012 \div 20.82509) = .41032$. This process continues until we converge to a solution of approximately .41147. Notice that the rate of convergence in this example is much faster when using the Newton-Raphson Method than when using the Method of Bisection.

Equation for f: $S_0N(d_1)-Xe^{-rT}N(d_2)$

$r = 0.1$ $S_0 = 75$ $X = 80$ $c_0 = 8.20$ $T = 0.5$ $\sigma_0 = 0.6$

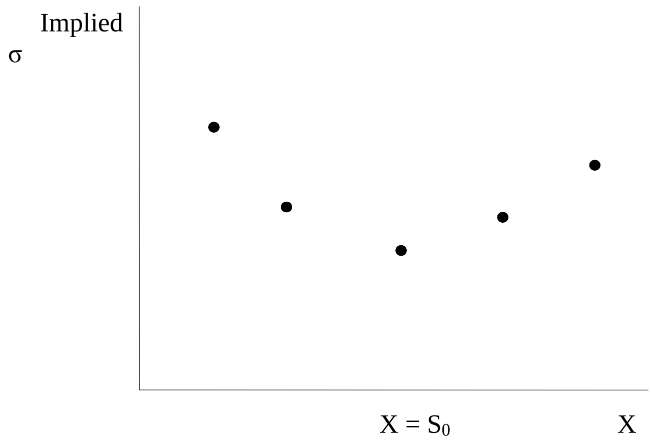
n	**σ_n**	**$f'(\sigma_n)$**	**$d_1(\sigma_n)$**	**$d_2(\sigma_n)$**	**N(d_1)**	**N(d_2)**	**$f(\sigma_n)$**
1	.60000	20.82509	.177864	-.2464	.5705853	.402686	3.95012
2	.41032	21.06194	.094961	-.19518	.5378271	.422626	-0.02415
3	.41147	21.06085	.095506	-.19544	.5380436	.422522	0.00000

[2] See Section 7.5 for additional details and discussion concerning vega, one of the "Greeks," and Exercise 7.7 for its derivation.

Table 2: The Newton-Raphson Method and Implied Volatilities

Smiles, Smirks and Aggregating Procedures

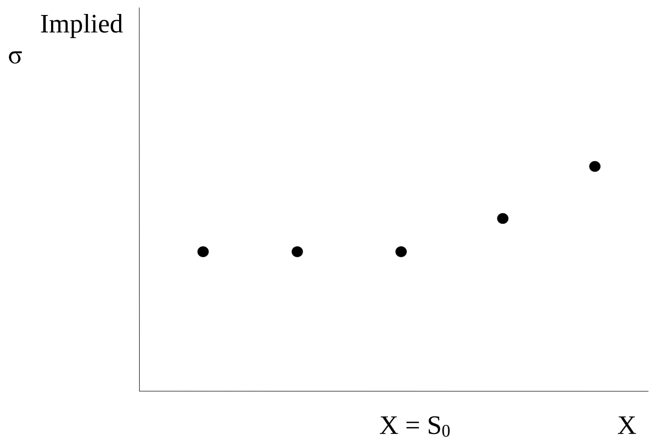
We see that with an appropriate iteration methodology, solving for implied volatility is not a difficult matter. However, another difficulty arising with implied variance estimates results from the fact that there will typically be more than one option trading on the same stock. However, what if the short- and long-term uncertainty of a stock differ? Or, what if options with different strike prices disagree on the same underlying stock volatility? Both of these inconsistencies regularly occur. This latter effect in which implied volatilities vary with respect to option exercise prices is sometimes known as the smile or smirk effect. See Figure 1 that depicts a smile effect for 5 options on a single stock and Figure 2 that depicts a smirk effect for a series of options on a second stock.[3] Each option's market price will imply its own underlying stock variance, and these variances are likely to differ. How might we use this conflicting information to generate the most reliable variance estimate? Each of our implied variance estimates is likely to provide some information, yet has the potential for having measured with error.



Options with high or low exercise prices relative to the current underlying stock price produce higher implied volatilities.

Figure 1: The Volatility Smile: Implied Volatility Given S_0

[3] The "smile effect" pertains to the empirical finding that options with very high and very low exercise prices relative to current underlying security prices produce high implied volatilities relative to options trading at or near the money. The "smirk effect" occurs when only either high or low exercise prices produce higher implied volatilities.



Options with high exercise prices relative to the current underlying stock price produce higher implied volatilities.

Figure 2: The Volatility Smirk: Implied Volatility Given S_0

Settling these implied volatility problems is largely an empirical or practitioner issue.[4] There is an empirical literature that focuses on how to use the information implied by two or more options to determine the appropriate volatility forecast. For example, we can preserve much of the information from each of our estimates and eliminate some of our estimating error if we use for our own implied volatility a value based on an average of all of our estimates. However, because volatility might be expected to vary over time, one should average only those variances implied by options with the same terms to expiration. Consider the following two methodologies for averaging implied standard deviation estimates:

1. *Simple average*: Here, the final standard deviation estimate is simply the mean of the standard deviations implied by the market prices of the calls.
2. *Average based on price sensitivities to* σ: Calls that are more sensitive to σ as indicated by $\partial c/\partial\sigma$ are more likely to imply a correct standard deviation estimate. Suppose we have n calls on a stock, and each call price c_j has an implied stock standard deviation σ_j. Each call price will have a sensitivity (vega; discussed in the next section) to its implied underlying stock standard deviation $\partial c_j/\partial\sigma_j$. The n option sensitivities can be summed, and a weighted average standard deviation estimate for the underlying stock based on its n options can be computed where the weight w_i associated with the implied standard deviation estimate for call option i is:

$$w_i = \frac{\frac{\partial c_{0,i}}{\partial \sigma_i}}{\sum_{j=1}^{n} \frac{\partial c_{0,j}}{\partial \sigma_j}} = \frac{S_{0,i}\sqrt{T}\frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}}}{\sum_{j=1}^{n} S_{0,j}\sqrt{T}\frac{e^{\frac{-d_{1,j}^2}{2}}}{\sqrt{2\pi}}}.$$

Thus, the final standard deviation estimate for a given stock based on all of the implied standard deviations from each of the call prices is:

[4] There are a small number of studies that suggest that historical volatilities contain useful information not contained in implied volatilities, and might even, in some instances, be better predictors of future volatility (See, for example, Canina and Figlewski [1993]).

$$\sigma = \sum_{i=1}^{n} w_i \sigma_i$$

$$\sigma = \sum_{i=1}^{n} w_i \sigma_i$$

While these aggregating procedures may provide useful information for volatility estimates, it might be useful to make entirely different sets of assumptions for option analysis, For example, while the Black-Scholes model assumes constant volatility, in Chapter 8, we will briefly discuss the assumption of stochastic volatility. More generally, we can allow volatility to be any function of time. In Chapter 2, we discussed jumps and Poisson processes. Black-Scholes assumes Brownian motion; this assumption can be relaxed to also allow for stock price jumps. We can also allow for time-varying interest rates, as we will discuss in Chapter 8. All of these assumption adjustments have the potential to explain or reduce so-called smile or smirk effects.

D. Empirical Evidence

Does the Black-Scholes model do a reasonable job explaining how investors price options? Numerous empirical tests yield evidence on this issue, generally finding that the model does work quite well explaining the pricing structure of stock options. However, the tests do reveal some biases.

<u>The Black and Scholes Study</u>

Black and Scholes [1972] conducted the first empirical test of their model. They collected over-the-counter price data on 2039 calls and 3052 straddles market securities on 545 underlying stocks from 1966 to 1969. In these markets, options were dividend protected (the option exercise price decreased on ex-dividend dates). They evaluated options whenever possible (if they were traded on a given day, though secondary markets were very thin). In most instances, they were unable to use market prices and had to value options based on their own model for the purpose of updating their hedges. In their hypothetical portfolios, bought calls if they were undervalued and sold the overvalued ones. They formed hedge portfolios with calls and their underlying shares based on their computed deltas. The portfolios that they formed are categorized by call option transaction as follows:

1. Buy all calls at market prices
2. Buy all calls at model prices
3. Buy undervalued calls and sell overvalued calls at model prices
4. Buy undervalued calls and sell overvalued calls at market prices

Again, the calls were combined into hedge portfolios with underlying shares. Excess returns as of option expiry on their hedge portfolios were defined as:

$$\Delta V_H - V_H r_F \Delta t$$

$$\Delta V_H = \left[\Delta C - \frac{\partial C}{\partial S}\Delta S\right] - \left[c - \frac{\partial C}{\partial S} S\right] r_f \Delta t \;\; where \;\; \frac{\partial C}{\partial S} = N(d_1) = hedge\ ratio$$

Excess hedge portfolio returns should equal zero if the Black-Scholes model explains market pricing. Black and Scholes updated hedge ratios as calls were amortized and as underlying share prices changed.

Black and Scholes found that use of holding period (ex post) variances yielded insignificant holding period hedge portfolio profits when transacting at model-computed prices with zero transactions costs. However, they also found that using historical (ex ante) variances to value calls yielded significant positive profits. Thus, the market uses more than historical variances to estimate true variances. Returns were uncorrelated with the market. On the other hand, transactions costs eliminated these profits.

In the absence of transactions costs, Black and Scholes found that portfolios 1 and 2 did not result in significant returns, indicating no consistent under- or over-valuation by either the market or the model. Portfolio 3 yielded negative returns; portfolio 4 yielded positive returns. This indicates that market prices contain information not incorporated by the model, nonetheless, one can still earn profits before transactions costs employing Black Scholes because the model contains some information not incorporated in market prices. In any case, Black and Scholes explained return results for portfolios 3 and 4 as follows: The market underestimates variances for high-risk stocks, yet the forecasts based on historical data are actually too high. Hence, model prices for high-risk stocks are too high; market prices are too low. The opposite is true for low - risk stocks. The high-risk stock effects will outweigh the low-risk stock effects.

There are a few concerns regarding the Black-Scholes tests that are worth noting. First, perfect continuous hedges are impossible. However, Black and Scholes updated portfolios daily and argue that their returns are uncorrelated with the market. Hence, the hedging errors can be diversified away. Second, since OTC options prices are observed only when they are created or expire, Black Scholes had to create artificial prices from their model daily. Option values are needed to determine hedge ratios. However, final portfolio returns were determined at option expiration. In conclusion, Black and Scholes found that their model seemed to work best for medium maturity at-the-money calls.

The Galai and Bhattacharya Studies

Galai [1977] used daily CBOE data from 4-26-1973 to 11-30-1973. He adjusted hedges on a daily basis with options transactions, not stock transactions like Black Scholes. Galai found that the Black Scholes Model resulted in significant excess returns before transactions costs but that transactions costs (at 1%) exceeded excess returns. His results held given changes in variances and risk-free rates, but that higher dividend yield stock options yielded lower profits. This result may simply reflect that Black Scholes assumes no dividends. Galai further found that model specification deviations led to worse performance and that tests of spread strategies led to results similar to those of the individual options.

Bhattacharya [1980] also found that Black Scholes Model prices were appropriate most of the time. Bhattacharya structured hedge portfolios based on simulated Black-Scholes values and found only one significant systematic case of mis-pricing - with at the money options whose prices were too high at expiration.

Smiles and Smirks

MacBeth and Merville [1979] studied estimated implied volatilities on all options on six stocks during the period 1975-76 and found the following:

1. Black Scholes prices are too low for in the money options and are too high for out of the money options. Mispricing worsens as the option is further in or out of the money.
2. Mispricing in (1) above worsens as the time to expiration of the option increases.

Thus, the model obtains higher implied volatilities for short-term in-the-money options than for long-term out-of-the-money options. Although Blattberg and Gonedes [1974] had already found that option implied volatilities are time-variant, if the Black-Scholes Model is correct, implied volatilities should be invariant with respect to the extent to which the option is in- or out-of-the-money. Thus, the results imply that in the money options are overpriced in the market relative to out of the money options if we accept Black-Scholes as correct. Market prices become more in line with Black-Scholes as expiration draws near.

MacBeth and Merville [1980] compared the Black Scholes Model to the Constant Elasticity of Variance Model with mixed results, generally finding that the Constant Elasticity of Variance formula worked slightly better than Black-Scholes.

Put-Call Parity

Klemkosky and Resnick [1979] tested the Put-Call Parity relation on 606 hedges, considering both short and long hedge portfolios and assuming non-stochastic dividends. They removed from their data set those options that were likely to be exercised early. Their results were consistent with the Put-Call Parity Theorem, finding that profits were generally within the bounds associated with transactions costs. Gould and Galai [1974] also test put-call parity:

$$c_0 - p_0 - S_0 + B_0 \leq 0$$

or, given transactions costs (TC):

$$c_0 - p_0 - S_0 + B_0 \leq TC$$

Gould and Galai [1974] compute call, put, stock and bond prices to infer transactions costs. They find that the inferred transactions costs were lower than actual transactions costs, implying that the market must be efficient with respect to put call parity. However, their tests (as well as the tests of Stoll [1969]) may be biased because of dividend protection.

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Exercises

1. Demonstrate that the definition given for self-financing in property (7) is equivalent to the condition that over any infinitesimal time interval from *t* to *t* + *dt*, the change in the value of the portfolio resulting purely from the transactions $d\gamma_{s,t}$ and $d\gamma_{b,t}$ that are executed over this time interval must equal zero.

2. Suppose that a self-financing portfolio includes a single short position in a call option with exercise price X. This single short position remains constant over time t. The portfolio also has a long position in $\gamma_{s,t}$ shares of stock that varies over time t along with $\gamma_{b,t}$ short positions in bonds (face value = X) that also vary over time. The bonds will be paid off by the exercise money realized if and when the call is exercised at time T.

a. Write an expression that gives the portfolio value V_T at time T.

b. Suppose that at time T, the value of the stock exceeds the exercise price X of the call. What will be $\gamma_{s,T}$ and $\gamma_{b,T}$?

c. Suppose that at time T, the value of the stock is less than the exercise price X of the call. What will be $\gamma_{s,T}$ and $\gamma_{b,T}$?

3. Assuming a Black-Scholes environment, evaluate calls and puts for each of the following European stock option series:

Option 1	Option 2	Option 3	Option 4
T = 1	T = 1	T = 1	T = 2
S = 30	S = 30	S = 30	S = 30
σ = .3	σ = .3	σ = .5	σ = .3
r = .06	r = .06	r = .06	r = .06
X = 25	X = 35	X = 35	X = 35

4. Evaluate each of the European options in the series on ABC Company stock assuming a Black-Scholes environment. Current market prices for each of the options are listed in the table. Determine whether each of the options in the series should be purchased or sold at the given market prices. The current market price of ABC stock is 120, the August options expire in nine days, September options in 44 days and October options in 71 days. The stock variances prior to expirations are projected to be .20 prior to August, .25 prior to September, and .20 prior to October. The treasury bill rate is projected to be .06 for each of the three periods prior to expiration. Convert the number of days given to fractions of 365-day years, as we shall assume that trading occurs 365 days per year.

CALLS

X	AUG	SEP	OCT
110	9.500	10.500	11.625
115	4.625	7.000	8.125
120	1.250	3.875	5.250
125	.250	2.125	3.125
130	.031	.750	1.625

σ = .20 FOR AUG
σ = .25 FOR SEP
σ = .20 FOR OCT
r =.06
S = 120

PUTS

X	AUG	SEP	OCT

110	.031	.750	1.500
115	.375	1.750	2.750
120	1.625	6.750	4.500
125	5.625	6.750	7.875
130	10.625	10.750	11.625

Exercise prices for 15 calls and 15 puts are given in the left columns. Expiration dates are given in column headings and current market prices are given in the table interiors.

5. Use put-call parity and the Black-Scholes call pricing model to verify the following in a Black-Scholes environment:

$$p_0 = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$$

6. Emu Company stock currently trades for $50 per share. The current riskless return rate is .06. Under the Black-Scholes framework, what would be the standard deviations implied by six-month (.5 year) European calls with current market values based on each of the following striking prices? That is, with market prices of calls taken as given and equal to Black-Scholes estimates, what standard deviation estimates in Black-Scholes models would yield call values equal to market values in each of the following scenarios?

a. X = 40; c_0 = 11.50
b. X = 45; c_0 = 8.25
c. X = 50; c_0 = 4.75
d. X = 55; c_0 = 2.50
e. X = 60; c_0 = 1.25

7. Cannondale Company stock is currently selling for $40 per share. Its historical standard deviation of returns is .5. The one-year Treasury bill rate is currently 5%. Assume that all of the standard Black-Scholes Option Pricing Model assumptions hold.

a. What is the value of a put on this stock if it has an exercise price of $35 and expires in one year?
b. What is the implied probability that the value of the stock will be less than $30 in one year?

Solutions

1. By the general product rule for stochastic processes in Section 6.1.1, the change in the value of the portfolio equals:

$$dV_t = d(\gamma_{s,t}S_t) + d(\gamma_{b,t}B_t) = \gamma_{s,t}dS_t + S_t d\gamma_{s,t} + dS_t d\gamma_{s,t} + \gamma_{b,t}dB_t + B_t d\gamma_{b,t} + dB_t d\gamma_{b,t}$$
$$= (S_t + dS_t)d\gamma_{s,t} + (B_t + dB_t)d\gamma_{b,t} + \gamma_{s,t}dS_t + \gamma_{b,t}dB_t.$$

The infinitesimal transactions are buying or shorting $d\gamma_{s,t}$ shares of the stock at a price of S_t+dS_t per share and $d\gamma_{b,t}$ units of the bond at price of B_t+dB_t per unit. Thus, the change in value resulting from these transactions is $(S_t+dS_t)d\gamma_{s,t}+(B_t+dB_t)d\gamma_{b,t}$. From the equation above, we see that $dV_t = \gamma_{s,t}dS_t + \gamma_{b,t}dB_t$ if and only if $(S_t+dS_t)d\gamma_{s,t}+(B_t+dB_t)d\gamma_{b,t} = 0$ as we set out to prove.

2. a. $V_T = -c_T + \gamma_{s,T}S_T + \gamma_{b,T}B_T = -MAX[S_T - X, 0] + \gamma_{s,T}S_T + \gamma_{b,T}X = 0$

b. Since $c_T = S_T - X$ when S_T exceeds X, then $\frac{\partial c_T}{\partial S} = 1$. In the Black-Scholes derivation we showed that we must choose $\gamma_{s,T} = \frac{\partial c_T}{\partial S} = 1$. Solving for $\gamma_{b,t}$ in the equation: $-MAX[S_T - X, 0] + \gamma_{s,T}S_T + \gamma_{b,T}X = -[S_T - X, 0] + 1 \times S_T + \gamma_{b,T}X = 0$, we find that $\gamma_{b,t} = -1$. This strategy ensures that the portfolio is self-financing at time T and has a time T value equal to zero: $V_T = -[S_T - X] + \gamma_{s,T}S_T + \gamma_{b,T}X = 0$.

c. Since $c_T = 0$ when S_T is less than X, then $\frac{\partial c_T}{\partial S} = 0$. In the Black-Scholes derivation we showed that we must choose $\gamma_{s,T} = \frac{\partial c_T}{\partial S} = 0$. Solving for $\gamma_{b,t}$ in the equation: $-MAX[S_T - X, 0] + \gamma_{s,T}S_T + \gamma_{b,T}X = -0 + 0 \times S_T + \gamma_{b,T}X = 0$, we find that $\gamma_{b,t} = 0$. This strategy ensures that the portfolio is self-financing at time T and has a time T value equal to zero: $V_T = -0 + \gamma_{s,T}S_T + \gamma_{b,T}X = 0$.

3. The options are valued with the Black-Scholes Model in a step-by-step format in the following table:

	OPTION 1	OPTION 2	OPTION 3	OPTION 4
d(1)	.957739	-.163836	.061699	.131638
d(2)	.657739	-.463836	-.438301	-.292626
N[d(1)]	.830903	.434930	.524599	.552365
N[d(2)]	.744647	.321383	.330584	.384904
Call	7.395	2.455	4.841	4.623
Put	0.939	5.416	7.803	5.665

4. Value the calls using the Black-Scholes Model:

$c_0 = S_0N(d_1) - Xe^{-rT}N(d_2)$
$d_1 = [\ln(S \div X) + (r + .5\sigma^2)T] \div \sigma\sqrt{T}$
$d_2 = d_1 - \sigma\sqrt{T}$

Thus, we will first compute d_1, d_2, $N(d_1)$, $N(d_2)$ for each of the calls; then we will compute each call's value. We will then use put-call parity to value each put. First find for each of the 15 calls values for d_1:

X	AUG	SEP	OCT
110	2.833394	1.129163	1.162841
115	1.417978	.617046	.658904
120	.062811	.126728	.176418
125	-1.237028	-.343571	-.286369
130	-2.485879	-.795423	-.731003

Next, find for each of the 15 calls values for d_2:

X	AUG	SEP	OCT
110	2.801988	1.042362	1.074632
115	1.386572	.530245	.570695
120	.031405	.039928	.088209
125	-1.268433	-.430371	-.374578
130	-2.517284	-.882222	-.819212

Now, find $N(d_1)$ for each of the 15 calls:

X	AUG	SEP	OCT
110	.997697	.870585	.877553
115	.921901	.731398	.745021
120	.525041	.550422	.570017
125	.108038	.365584	.387298
130	.006462	.213184	.232389

Next, determine $N(d_2)$ for each of the 15 calls:

X	AUG	SEP	OCT
110	.997461	.851378	.858730
115	.917214	.702029	.715897
120	.512527	.515925	.535145
125	.102322	.333463	.353987
130	.005913	.188828	.206333

Now use $N(d_1)$ and $N(d_2)$ to value the calls and put-call parity to value the puts.

CALLS

X	AUG	SEP	OCT
110	10.165	11.494	11.942
115	5.305	7.616	<u>8.030</u>
120	1.593	4.586	<u>4.930</u>
125	<u>.193</u>	2.488	<u>2.741</u>
130	<u>.008</u>	1.211	<u>1.375</u>

PUTS

X	Aug	Sep	Oct
110	<u>0.003</u>	<u>.701</u>	<u>0.666</u>
115	<u>0.134</u>	1.787	<u>1.695</u>
120	<u>1.415</u>	<u>3.721</u>	<u>3.537</u>
125	<u>5.009</u>	<u>6.587</u>	<u>6.290</u>
130	<u>9.816</u>	<u>10.274</u>	<u>9.866</u>

The options whose values are underlined are overvalued by the market; they should be sold. Other options are undervalued by the market; they should be purchased.

5. Put-call parity states the first relation generally, and the second in a Black-Scholes environment:

$$p_0 = c_0 + Xe^{-rT} - S_0$$

$$p_0 = S_0N(d_1) - \frac{X}{e^{rT}}N(d_2) + Xe^{-rT} - S_0$$

With some algebra, and given the symmetry of the normal distribution about its mean, we rewrite as follows:

$$p_0 = S_0(N(d_1) - 1) - \frac{X}{e^{rT}}(N(d_2) - 1) = Xe^{-rT}N(-d_2) - S_0N(-d_1)$$

6. We need to find the roots of the equation:

$$f(\sigma) = S_0N(d_1) - Xe^{-rT}N(d_2) = 0$$

with $S_0 = 50$, $r = .06$, and $T = .5$. We let X and c_0 assume the values given in parts a through e, respectively. One can merely substitute for variance σ by testing the equation to see if $f(\sigma)$ is approximately equal to zero for many choices of σ until one finds a good approximation. Otherwise, one can use either the Bisection Method or the Newton Raphson Method. The values for σ are obtained applying one of these two methods through a process of substitution and iteration until the desired accuracy is obtained.

Implied volatilities are given as follows:

a. X = 40; σ = .2579
b. X = 45; σ = .3312
c. X = 50; σ = .2851
d. X = 55; σ = .2715
e. X = 60; σ = .2704

7. a. $d_1 = .6171$; $d_2 = .1171$; $N(d_1) = .7314$; $N(d_2) = .5466$
$c_0 = 11.06$; with put-call parity: $p_0 = 4.35$
b. Use X=30; $d_1 = .9254$; $d_2 = .4254$; $N(d_2) = .6647$
In section 5.4.3 we showed that $P(S_1 < 30) = 1 - P(S_1 > 30) = 1 - N(d_2) = .3353$.

Appendix 10.A: Solving the Black-Scholes Differential Equation

The following is the Black-Scholes differential equation:

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2},$$

generalized somewhat to value any applicable derivative *V* rather than just a call *c*. In order to solve for *V* = *V*(*S*,*t*), the value of the derivative, one also needs to be given a boundary value. This means that the value of the derivative must be known at some fixed time *T*; that is, *V*(*S*,*T*) is a given function of *S* and *T*. With this information, one can solve for the value *V*(*S*,*t*) at any time *t*. Here are the ideas of the proof in a nutshell. One changes variables from *V, S,* and *t* to the new variables *u*,*y,* and τ in two separate procedures so that the Black-Scholes Differential Equation simplifies to the differential equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}.$$

This equation is the classical heat equation, whose solution is well-known. One then changes back to the original variables to obtain the solution. This solution will be in integral form. We will then solve for the special case of a European call. In this case, it will turn out that the integral solution will be able to be expressed in terms of the cumulative normal density function. Now for the details. We start the solution process by first changing variables to replace t, S and V with τ, *y*, and *v*:

$$\tau = \frac{1}{2}\sigma^2(T-t)\ ;\ t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$$

$$y = \ln\frac{S}{X}\ ;\ S = Xe^y$$

$$v\big(y(S), \tau(t)\big) = \frac{C(S,t)}{X}; V(S,t) = Xv\left(\ln\left(\frac{S}{X}\right), \frac{1}{2}\sigma^2(T-t)\right).$$

The following derivatives follow from the first two left-hand equations above:

$$\frac{\partial \tau}{\partial t} = -\frac{1}{2}\sigma^2\ ; \frac{\partial \tau}{\partial S} = 0; \frac{\partial y}{\partial t} = 0\ ; \frac{\partial y}{\partial S} = \frac{1}{S}$$

Next, we use the Chain Rule to rewrite the Black-Scholes equation in terms of *v* and its partial derivatives with respect to *y* and τ:

$$\frac{\partial V}{\partial S} = \frac{\partial (Xv)}{\partial y}\frac{\partial y}{\partial S} + \frac{\partial (Xv)}{\partial \tau}\frac{\partial \tau}{\partial S} = X\frac{\partial v}{\partial y}\frac{1}{S} + 0 = \frac{X}{S}\frac{\partial v}{\partial y}$$

$$\frac{\partial V}{\partial t} = \frac{\partial (Xv)}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial (Xv)}{\partial \tau}\frac{\partial \tau}{\partial t} = 0 - X\frac{\partial v}{\partial \tau}\frac{1}{2}\sigma^2 = -\frac{1}{2}\sigma^2 X\frac{\partial v}{\partial \tau}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial\left(\frac{X}{S}\frac{\partial v}{\partial y}\right)}{\partial S} = \frac{\partial\left(\frac{X}{S}\right)}{\partial S}\frac{\partial v}{\partial y} + \frac{X}{S}\frac{\partial\left(\frac{\partial v}{\partial y}\right)}{\partial S} = -\frac{X}{S^2}\left(\frac{\partial v}{\partial y}\right) + \frac{X}{S}\left(\frac{\partial^2 v}{\partial y^2}\ \frac{\partial y}{\partial S}\right) = \frac{X}{S^2}\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\right)$$

Now, we will substitute these equations into the Black-Scholes differential equation:

$$-\frac{1}{2}\sigma^2 X\frac{\partial v}{\partial \tau} = rXv - rS\frac{X}{S}\frac{\partial v}{\partial y} - \frac{1}{2}\sigma^2 S^2\frac{X}{S^2}\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\right)$$

Simplify and divide both sides by $\frac{1}{2}\sigma^2 X$:

$$-\frac{\partial v}{\partial \tau} = \frac{2r}{\sigma^2}v - \frac{2r}{\sigma^2}\frac{\partial v}{\partial y} - \left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\right)$$
$$\frac{\partial v}{\partial \tau} = \frac{-2r}{\sigma^2}v + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}$$

However, our equation still includes terms involving v and $\partial v/\partial y$, and we still need to employ an additional changes of variables in order to obtain the desired classic heat equation . We will again employ a change of variables, writing v as a function of u:

$$v(y,\tau) = e^{\alpha y+\beta\tau}u(y,\tau),$$

where α and β are constants to be chosen shortly. We differentiate as follows:

$$\frac{\partial v}{\partial \tau} = \beta e^{\alpha y+\beta\tau}u + e^{\alpha y+\beta\tau}\frac{\partial u}{\partial \tau}$$
$$\frac{\partial v}{\partial y} = \alpha e^{\alpha y+\beta\tau}u + e^{\alpha y+\beta\tau}\frac{\partial u}{\partial y}$$
$$\frac{\partial^2 v}{\partial y^2} = \alpha^2 e^{\alpha y+\beta\tau}u + 2\alpha e^{\alpha y+\beta\tau}\frac{\partial u}{\partial y} + e^{\alpha y+\beta\tau}\frac{\partial^2 u}{\partial y^2}$$

We will substitute these three derivatives and the expression for v into our already transformed Black-Scholes equation:

$$\beta e^{\alpha y+\beta\tau}u + e^{\alpha y+\beta\tau}\frac{\partial u}{\partial \tau}$$
$$= \frac{-2r_f}{\sigma^2}e^{\alpha y+\beta\tau}u + \left(\frac{2r}{\sigma^2} - 1\right)\left(\alpha e^{\alpha y+\beta\tau}u + e^{\alpha y+\beta\tau}\frac{\partial u}{\partial y}\right) + \alpha^2 e^{\alpha y+\beta\tau}u$$
$$+ 2\alpha e^{\alpha y+\beta\tau}\frac{\partial u}{\partial y} + e^{\alpha y+\beta\tau}\frac{\partial^2 u}{\partial y^2}$$

Divide by $e^{\alpha y+\beta\tau}$ to obtain:

$$\beta u + \frac{\partial u}{\partial \tau} = \frac{-2r}{\sigma^2} u + \left(\frac{2r}{\sigma^2} - 1\right)\left(\alpha u + \frac{\partial u}{\partial y}\right) + \alpha^2 u + 2\alpha \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2},$$

Combine like terms to obtain our second revision of the Black-Scholes differential equation:

$$\frac{\partial u}{\partial \tau} = \left[\frac{-2r}{\sigma^2} + \left(\frac{2r}{\sigma^2} - 1\right)\alpha + \alpha^2 - \beta\right] u + \left(\frac{2r}{\sigma^2} - 1 + 2\alpha\right)\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}$$

Observe that if we choose

$$\alpha = \frac{1}{2} - \frac{r}{\sigma^2},$$

then the coefficient for the term $\partial u/\partial y$ will equal zero, and if we furthermore choose

$$\beta = \frac{2r}{\sigma^2} - \left(\frac{2r}{\sigma^2} - 1\right)\alpha - \alpha^2 = -\frac{r^2}{\sigma^4} - \frac{r}{\sigma^2} - \frac{1}{4},$$

then the coefficient for the term u will equal zero. With these choices for the constants α and β, we obtain our first main objective:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}.$$

Next, we solve this classic heat equation. Recall that we also need to be given the initial boundary condition. So, we need to know how to transform the boundary condition $V(S,T)$ to the function u in terms of y and the proper choice of τ. Since $\tau = \frac{1}{2}\sigma^2(T - t)$, then when $t = T$, $\tau = 0$. Since $(S,T) = Xv\left(ln\left(\frac{S}{X}\right), 0\right) = Xv(y, 0)$, $v(y,0) = e^{\alpha y}u(y,0)$, and $S = Xe^y$, then

$$u(y, 0) = \frac{V(Xe^y, T)}{Xe^{\alpha y}}.$$

The solution of the heat equation $u(y,\tau)$ in terms of the boundary condition $u(y,0)$ is

(32) $$u(y, \tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} u(x, 0) e^{-\frac{(y-x)^2}{4\tau}}\, dx.$$

To prove that this solution is correct, we find[5]

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[5] The differentiation operation was brought inside the integral sign, which is justified as long as the initial condition $u(x,0)$ is a reasonably nice function. This will always be the case for any initial conditions encountered in this text.

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \tau}\left[\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)e^{-\frac{(y-x)^2}{4\tau}}\,dx\right] = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} u(x,0)\frac{\partial}{\partial \tau}[\frac{1}{\sqrt{\tau}}e^{-\frac{(y-x)^2}{4\tau}}]dx$$

$$= \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} u(x,0)[\frac{-1}{2\tau^{3/2}} + \frac{(y-x)^2}{4\tau^{5/2}}]e^{-\frac{(y-x)^2}{4\tau}}\,dx$$

$$= \frac{-1}{2\tau\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)[1 - \frac{(y-x)^2}{2\tau}]e^{-\frac{(y-x)^2}{4\tau}}\,dx,$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)\frac{\partial^2}{\partial y^2}[e^{-\frac{(y-x)^2}{4\tau}}]dx = \frac{-1}{2\tau\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)\frac{\partial}{\partial y}[(y-x)e^{-\frac{(y-x)^2}{4\tau}}]dx$$

$$= \frac{-1}{2\tau\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)[1 - \frac{(y-x)^2}{2\tau}]e^{-\frac{(y-x)^2}{4\tau}}\,dx = \frac{\partial u}{\partial \tau}$$

as we wished to show. We also must show that the solution approaches *u*(*y*,*0*) in the limit as $\tau \to 0$. So, we need to show that[6]

$$\lim_{\tau\to 0}\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)e^{-\frac{(y-x)^2}{4\tau}}\,dx = u(y,0).$$

Make the change of variables $z = \frac{x-y}{\sqrt{2\tau}}$, so that $x = y + z\sqrt{2\tau}$, and $dx = dz\sqrt{2\tau}$. The left hand limit above becomes

$$\lim_{\tau\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(y + z\sqrt{2\tau}, 0)e^{-\frac{z^2}{2}}\,dz.$$

We will assume that the boundary condition grows no faster than an exponential of the form $Ae^{B|z|}$ where *A* and *B* are positive constants. This is a reasonable assumption since no real-life security will grow faster than such a rate. Thus, the entire function inside the integral gets small very rapidly once |*z*| gets large. So, most of the contribution to the value of the integral occurs in the range of integration from –*N* to *N* with *N* a large positive number. We can closely approximate the limit above by

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[6] The function $\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{x^2}{4\tau}}$ is an example of what is known as an approximate identity, and the integral is an example of what is called the convolution of the approximate identity with the function *u*(*x*,*0*). It is a well-known theorem in mathematics that in the limit as τ approaches 0, the convolution above approaches *u*(*y*,*0*). However, we will prove this result for our special case.

$$\lim_{\tau\to 0}\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} u(y+z\sqrt{2\tau})e^{-\frac{z^2}{2}}\,dz.$$

The quantity $y + z\sqrt{2\tau}$ will vary from $y - N\sqrt{2\tau}$ to $y + N\sqrt{2\tau}$ as *s* ranges over the limits of integration from –*N* to *N.* As $\tau \to 0$ and eventually gets much smaller than $1/N^2$, the quantity $y + z\sqrt{2\tau}$ will be approximately equal to *y* as *z* ranges over the limits of integration from –*N* to *N*. Thus, $u(y + z\sqrt{2\tau}) \approx u(y)$ over this range of integration as long as *u* is a continuous function. This means that

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} u(y+z\sqrt{2\tau})e^{-\frac{z^2}{2}}\,ds \approx \frac{1}{\sqrt{2\pi}}\int_{-N}^{N} u(y)e^{-\frac{z^2}{2}}\,dz = u(y)\left[\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} e^{-\frac{z^2}{2}}\,dz\right].$$

But, the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} e^{-\frac{z^2}{2}}\,dz \approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz = 1,$$

since *N* is very large and the right-hand integral is the total area under the standard normal curve. The approximations above get better and better as $N \to \infty$ and $\tau \to 0$. We conclude that

$$\lim_{\tau\to 0}\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u(x,0)e^{-\frac{(y-x)^2}{4\tau}}\,dx = u(y,0),$$

as we wished to show.

Our result allows us to find the value of any derivative that is based on an underlying security *S* that follows a Brownian motion process with drift. One only needs to substitute the specific boundary conditions for the derivative, to obtain its value at any time *t.* As an illustration, we will price a European call option. For a call option, it is customary to denote its value at time *t* in terms of the price of the stock *S* by *c*(*S*,*t*) rather than *V*(*S*,*t*). Suppose that *X* is its exercise price, and *T* is its expiration date. If *S* >*X* at expiration time *T*, then the call option will be worth *S*-*X*, since the option holder can buy the stock for *X* and sell it for *S.* If S ≤ *X* at time *T*, then the option holder should not exercise the option, making the value of the option simply equal to 0. This shows that the boundary condition for the call option at time *T* is *c*(*S*,*T*) = *MAX*(*S*-*X*,*0*). In order to value the price of the option at any other time *t*, it turns out that it is easier to first solve the problem for *u* as a function of *y* and τ. Afterwards, one can convert the solution to *c* as a function of *S* and *t.* We will need to find the boundary condition for the function *u.* Since $u(y,0) = \frac{c(Xe^y,T)}{Xe^{\alpha y}}$ as we showed earlier, then

$$u(y,0) = \frac{\text{MAX}(Xe^y - X, 0)}{Xe^{\alpha y}} = \text{MAX}\left(e^{(1-\alpha)y} - e^{-\alpha y}, 0\right).$$

Substituting this initial condition into our integral solution (32), we obtain:

$$u(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \max\left(e^{(1-\alpha)x} - e^{-\alpha x}, 0\right) e^{-\frac{(y-x)^2}{4\tau}}\, dx.$$

Observe that $\boldsymbol{e^{(1-\alpha)x}}$-$e^{-\alpha x} \geq 0$ means $e^{-\alpha x}(e^x\text{-}1) \geq 0$, or $e^x\text{-}1 \geq 0$, or $e^x \geq 1$, or $x \geq 0$, So, the solution becomes:

$$u(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} [e^{(1-\alpha)x} - e^{-\alpha x}] e^{-\frac{(y-x)^2}{4\tau}}\, dx.$$

The next few steps of the derivation involve some algebraic manipulation and a couple changes of variables in order to express the integral above in terms of the cumulative distribution function for the standard normal curve. Using completion of the square and some algebraic manipulations, we can write:

$$(1-\alpha)x - \frac{(y-x)^2}{4\tau} = -\frac{1}{4\tau}[x - (y + 2(1-\alpha)\tau)]^2 + (1-\alpha)y + (1-\alpha)^2\tau$$

and

$$-\alpha x - \frac{(y-x)^2}{4\tau} = -\frac{1}{4\tau}[x - (y - 2\alpha\tau)]^2 - \alpha y + \alpha^2\tau.$$

So, the solution now becomes:

$$u(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{(1-\alpha)y + (1-\alpha)^2\tau} \int_{0}^{\infty} e^{-\frac{1}{4\tau}[x-(y+2(1-\alpha)\tau)]^2}\, dx$$
$$- \frac{1}{\sqrt{4\pi\tau}} e^{-\alpha y + \alpha^2\tau} \int_{0}^{\infty} e^{-\frac{1}{4\tau}[x-(y-2\alpha\tau)]^2}\, dx.$$

In the first integral, make the change of variables, $z = \frac{1}{\sqrt{2\tau}}[x - (y + 2(1-\alpha)\tau)]$. In the second integral, make the change of variables, $z = \frac{1}{\sqrt{2\tau}}[x - (y - 2\alpha\tau)]$. This gives:

$$u(y,\tau) = e^{(1-\alpha)y+(1-\alpha)^2\tau} \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{\sqrt{2\tau}}(y+2(1-\alpha)\tau)}^{\infty} e^{-\frac{1}{2}z^2}\, dz - e^{-\alpha y+\alpha^2\tau} \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{\sqrt{2\tau}}(y-2\alpha\tau)}^{\infty} e^{-\frac{1}{2}z^2}\, dz.$$

Recall that the cumulative distribution function for the standard normal curve is

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}z^2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-y}^{\infty} e^{-\frac{1}{2}z^2}\, dz,$$

where the second equality follows because of the symmetry of the standard normal curve. Thus, the solution can be expressed as:

$$u(y,\tau) = e^{(1-\alpha)y+(1-\alpha)^2\tau} N\left(\frac{1}{\sqrt{2\tau}}[y + 2(1-\alpha)\tau]\right) - e^{-\alpha y+\alpha^2\tau} N\left(\frac{1}{\sqrt{2\tau}}[y - 2\alpha\tau]\right).$$

We are now ready to obtain the solution for price of the option $c(S,t)$. Since $c(S,t) = Xv(y,\tau) = Xe^{\alpha y+\beta\tau}u(y,\tau)$, $S = Xe^{y}$, and $y = ln(S/X)$, then

$$c = Se^{[\beta+(1-\alpha)^2]\tau} N\left(\frac{1}{\sqrt{2\tau}}\left[ln\left(\frac{S}{X}\right) + 2(1-\alpha)\tau)\right]\right) - Xe^{(\beta+\alpha^2)\tau} N\left(\frac{1}{\sqrt{2\tau}}\left[ln\left(\frac{S}{X}\right) + 2\alpha\tau)\right]\right).$$

Since $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$ and $\beta = -\frac{r^2}{\sigma^4} - \frac{r}{\sigma^2} - \frac{1}{4}$, it is easy to check that $\beta + (1-\alpha)^2 = 0$ and $\beta + \alpha^2 = -\frac{2r}{\sigma^2}$. Using these results and the fact that $\tau = \frac{1}{2}\sigma^2(T-t)$ results in

(33) $$c(S,t) = SN\left(\frac{ln\left(\frac{S}{X}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Xe^{-r(T-t)} N\left(\frac{ln\left(\frac{S}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Using the customary notation:

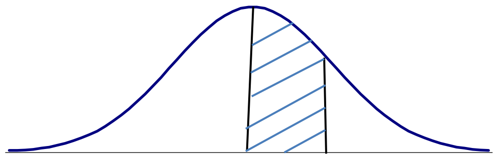
$$d_1 = \frac{ln\left(\frac{S_0}{X}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

we can finally express the solution at time $t = 0$ in the form

$$c_0 = S_0 N(d_1) - Xe^{-rT} N(d_2).$$

Appendix 10.B: z-table



The Normal Density Function
The z-Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0159	.0199	.0239	.0279	.0319	.0358
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0909	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2356	.2389	.2421	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2793	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3437	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3906	.3925	.3943	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4986	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

The areas given here are from the mean (zero) to *z* standard deviations to the right of the mean. To get the area to the left of *z*, simply add .5 to the value given on the table.