Chapter 9: Fundamentals of Stochastic Calculus

A. Stochastic Calculus: An Introduction

Here in Chapter 9, we will introduce and apply stochastic differential equations to modeling the pricing behavior of derivative instruments. In this chapter, we will introduce the theory and the methods of stochastic calculus. Although stochastic and ordinary calculus share many common properties, there are fundamental differences. The probabilistic nature of stochastic processes distinguishes them from the deterministic functions associated with ordinary calculus. Since stochastic differential equations so frequently involve Brownian motion, second order terms in the Taylor series expansion of functions become important, in contrast to ordinary calculus where they can be ignored. This attribute is reflected in Itô's Lemma, which is a powerful tool used to solve stochastic differential equations.

Arbitrage pricing (pricing securities relative to securities or portfolios producing identical payoff structures) can be accomplished numerous ways. A most useful feature of arbitrage pricing is that it does not require that we forecast security prices or even calculate risk premiums or expected returns. Such forecasts are, at best, unreliable and difficult to make. We use option pricing models such as the Black-Scholes or Binomial models that require no inputs for expected future security prices, risk premiums or expected returns (aside from the riskless security). However, such models do require knowledge of underlying security volatility, so it will be convenient to be able to delete references to expected security returns while maintaining information concerning volatility. This will entail creation of martingales.

Our analysis in this chapter will generally focus on continuous-time stochastic processes, though for development purposes, we will make references in certain sections to discrete-time processes for sake of simplicity. In this section, we will introduce stochastic differentiation and stochastic integration. In the section following, we will discuss Taylor series expansions and Itô's Lemma along with a number of applications and further discussions concerning stochastic integration. Much of the mathematics in this chapter will be applied to financial problems, mostly related to options in Chapter 10.

<u>Differentials of Stochastic Processes</u>

In many respects, differentials of stochastic processes mirror differentials of real-valued functions from ordinary calculus, sharing many of their properties. However, there are also important differences. In ordinary calculus, if X_t is a real-valued function of the real variable t, then the derivative X_t' exists for a large class of functions. If X_t is a stochastic process, then it is usually not possible to well-define the derivative of X_t with respect to t, at least for the class of stochastic processes that are relevant in finance. This is because normally X_t involves Brownian motion, and Brownian motion is not differentiable. Nevertheless, we can still study the differential of a stochastic process, which is the change of a stochastic process X_t resulting from a small change in t. Define the differential dX_t of a stochastic process X_t to be a quantity that satisfies the property

$$\lim_{dt\to 0}\frac{X_{t+dt}-X_t-dX_t}{dt}=0.$$

The differential dX_t is used to approximate $X_{t+dt} - X_t$, and any terms that approach zero after dividing by dt as $dt\to 0$ can be ignored. This is the same requirement that we have seen in ordinary calculus. In the following sub-sections, we will compare an example of a real-valued (ordinary) differential with that of a stochastic differential.

An Example of a Real-Valued Differential from Ordinary Calculus

Consider the (ordinary) real-valued function $X(t) = t^3$, with t being a real variable. Then:

$$X(t+dt) - X(t) = (t+dt)^3 - t^3 = t^3 + 3t^2dt + 3t(dt)^2 + (dt)^3 - t^3$$
$$= 3t^2dt + 3t(dt)^2 + (dt)^3.$$

Notice that $[3t(dt)^2 + (dt)^3]/dt = 3tdt + (dt)^2 \to 0$ as $dt \to 0$. This means that for small values of dt, $3t(dt)^2 + (dt)^3$ is much smaller than $3t^2dt$ and we can approximate $X(t+dt)$-$X(t)$ by $3t^2dt$. So, the differential of $X(t) = t^3$ is $dX = 3t^2dt$. As we saw in the review of the differential for real-valued continuously differential functions in Chapter 1, the differential of a function $X(t)$ equals the derivative of the function times dt. For both real-valued and stochastic differentials, one can always choose $X_{t+dt} - X_t$ itself as the differential dX_t, but it is often better to use another choice that is either more useful or leads to a simpler expression. We saw this illustrated earlier in this book, and we will see this situation again in the example below.

An Example of a Stochastic Differential

Next, consider the following stochastic process $X_t = tZ_t$, with Z_t being standard Brownian motion. We calculate that

$$X_{t+dt} - X_t = (t+dt)Z_{t+dt} - tZ_t = t(Z_{t+dt} - Z_t) + Z_{t+dt}dt$$

$$= t(Z_{t+dt} - Z_t) + Z_{t+dt}dt - Z_tdt + Z_tdt = t(Z_{t+dt} - Z_t) + (Z_{t+dt} - Z_t)dt + Z_tdt$$

$$= tdZ_t + dZ_tdt + Z_tdt.$$

Now, consider choosing $dX_t = tdZ_t + Z_tdt$ as a differential of X_t. With this choice, note that

$$\frac{X_{t+dt} - X_t - dX_t}{dt} = \frac{tdZ_t + dZ_tdt + Z_tdt - tdZ_t - Z_tdt}{dt} = \frac{dZ_tdt}{dt} = dZ_t.$$

Since $dZ_t = Z_{t+dt} - Z_t$, dZ_t is distributed normally with mean 0 and variance dt. Thus, with probability approaching 1, the values of dZ_t must be on the order of the standard deviation $\sqrt{dt}$. This means that the quotient above approaches 0 with probability 1 as $dt \to 0$. This confirms that $dX_t = tdZ_t + Z_tdt$ is a differential of X_t.

One of the most important applications of the differential is to evaluate integrals. Suppose a term in the integrand of an integral has the property that if one divides it by dt, the result will approach 0 as dt approaches 0. Then, the term itself will not contribute at all to the integral. This means that such a term can be ignored.

Next, we state three elementary but important properties of the differential:

1. *Linearity:* If X_t and Y_t are stochastic processes, and a and b are constants, then $d(aX_t + bY_t) = adX_t + bdY_t$.
2. *General Product Rule:* If X_t and Y_t are stochastic processes, then $d(X_tY_t) = X_tdYt +$

$Y_t dX_t + dX_t dY_t$.

3. *Special Product Rule:* Suppose that $dX_t = \mu_t dt + \sigma_t dZ_t$ and $dY_t = \rho_t dt$ where σ_t, μ_t, and ρ_t are stochastic processes, then $d(X_t Y_t) = X_t dYt + Y_t dX_t$.

First, the linearity property is stated in terms of a sum of just two processes, but it can be extended to any finite sum of constant multiples of stochastic processes. The special product rule is expressed in differential form. It has the same form as in ordinary calculus. However, we note that the result would be different if dY_t had a Brownian motion component.[1]

Stochastic Integration

In many respects, the theory of integration of stochastic processes mirrors integration of real-valued functions. If $f(x)$ is a real-valued continuous function defined on the interval $a \leq x \leq b$, recall that:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

Integration of a stochastic process X_t with respect to the real variable t can be defined in a way analogous to that of a real-valued continuous function in ordinary calculus. In order to take the integral X_t over the interval $a \leq t \leq b$, divide the interval into n equal parts so that $\Delta t = (b - a)/n$ and $t_i = a + i\Delta t$ for $i = 0, 1, \ldots, n$. Define

$$\int_a^b X_t dt = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} X_{t_i} \Delta t.$$

Observe that whenever this integral exists the result is a random variable, since it is a limiting sum of random variables.

Next, we define the integral of a stochastic process X_t with respect to a stochastic process Y_t. Divide the interval $a \leq t \leq b$ in the same way as above. Then

$$\int_a^b X_t dY_t = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} X_{t_i} \left(Y_{t_{i+1}} - Y_{t_i}\right).$$

Once again, whenever this integral exists, the result is a random variable, since it is a limiting sum of random variables. Notice that our first definition of an integral is a special case of the second by choosing $Y_t = t$ so that $dY_t = dt$. Although $Y_t = t$ is a real-valued function, it is also a special case of a stochastic process. This follows from the observation that for each fixed value of t, $X_t = t$ with probability 1. It is a random variable that can only take on one value; namely, t at time t.

[1] Proofs of these three properties are available in Knopf and Teall [2015].

Standard Brownian Motion Process

Consider the following particular case of stochastic integration that will arise frequently in finance. Suppose that σ_t and μ_t are stochastic processes (note that σ_t and μ_t can be chosen so that each takes on single constant values) and Z_t is standard Brownian motion. Then

$$\int_a^b (\mu_t dt + \sigma_t dZ_t) = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} \left[\mu_{t_i} \Delta t + \sigma_{t_i} \left(Z_{t_{i+1}} - Z_{t_i}\right)\right],$$

Where $\Delta t = (b - a)/n$, $t_0 = a$, $t_n = b$, and $t_{i+1} - t_i = \Delta t$ for $i = 1, 2, \ldots, n$. Observe that the expression on the right-hand side above is a sum of random variables, and if the limit exists, then the result will be a random variable. In fact, for a fairly general class of stochastic processes that limit above does exist and the result is a well-defined random variable.

Contrasting Integration of Real-valued Functions with Integration of Stochastic Processes

Consider the following two illustrations involving integration, the first a real-valued function and the second a stochastic process where Z_t is standard Brownian motion:

1. $\int_0^2 5dX_t$ with $X_t = t^2$.

$$\int_0^2 5dX_t = 5X_t|_0^2 = 5t^2|_0^2 = 5(2)^2 - 5(0)^2 = 20$$

2. $\int_0^2 5dZ_t$

$$\int_0^2 5dZ_t = 5Z_t|_0^2 = 5Z_2 - 5Z_0 = 5Z_2.$$

The solution in the first illustration is a number. The solution in the second illustration is a random variable since Z_2 is a normally distributed random variable, with mean 0 and variance 2. Thus, $5Z_2$ is a normally distributed random variable with mean 0 and variance $(5)^2(2) = 50$.

Suppose one keeps the lower limit of integration a constant and replaces the upper limit of integration with a variable. In the case of integration of real-valued functions, the result is a real-valued function. Analogously, in the case of integration of stochastic processes, the result is a stochastic process.

Elementary Properties of Stochastic Integrals

Integral of a Stochastic Differential:

If X_t is a stochastic process, the integral of a stochastic differential dX_t is simply:

$$\int_a^b dX_t = X_b - X_a.$$

Alternatively, if we define the notation $X_t|_a^b = X_b - X_a$ as is used in ordinary calculus, then we can write this property as:

$$\int_a^b dX_t = X_t|_a^b = X_b - X_a.$$

Linearity

If X_t, Y_t, U_t, and V_t are stochastic processes, and C and D are real numbers, then

$$\int_a^b (CU_t dX_t + DV_t dY_t) = C\int_a^b U_t dX_t + D\int_a^b V_t dY_t,$$

as long as each of the integrals are defined. This rule constitutes the familiar "sum" and "constant multiple" rules for integrals of real valued functions in calculus carrying over to stochastic processes.

Illustration

Consider the following illustration. Suppose that the price of a security follows an arithmetic Brownian motion process with drift (a deterministic increment at a constant rate over time): $dX_t = .06dZ_t + .02dt$. Further suppose that the price of the security at time zero equals 20: $X_0 = 20$. Now we seek to find X_t, the price of the security at time t: First observe that

$$\int_0^t dX_s = \int_0^t (.06dZ_s + .02ds).$$

By the first property, the left side equals:

$$\int_0^t dX_s = X_s|_0^t = X_t - 20,$$

By linearity and the first property, the right-side equals:

$$\int_0^t (.06dZ_s + .02ds) = (.06Z_s + .02s)|_0^t = .06Z_t + .02t - .06Z_0 = .06Z_t + .02t.$$

Setting these results equal and solving for X_t, we find that $X_t = 20 + .02t + .06Z_t$. The security price at time t equals its price at time zero plus .02 multiplied by the elapsed time plus .06 times the Brownian motion. The price is a normally distributed random variable with mean $20+.02t$ and variance $(.06)^2t$.

B. A Digression on Taylor Series Expansions

As preparation for our discussion on Itô's Lemma in the next section, we will briefly review Taylor series expansions, a basic topic in Calculus. We can use the Taylor series expansion to estimate the change in an infinitely differentiable function $y = y(t)$ as follows:

$$\Delta y = \frac{dy}{dt} \cdot \Delta t + \frac{1}{2} \cdot \frac{d^2 y}{dt^2} \cdot (\Delta t)^2 + ...$$

If Δt is small, higher order terms (involving Δt raised to powers 2 or greater) are negligible compared to terms just involving Δt. So we have the following approximation when Δt is small:

$$\Delta y = \frac{dy}{dt} \Delta t.$$

Taylor Series and the Differential Notation

We can also be express our Taylor series approximation using differential notation:

$$dy = \frac{dy}{dt} dt.$$

In particular, when evaluating integrals, terms involving Δt raised to powers 2 or larger can be dropped. Thus, for example:

$$y(T) - y(0) = \int_0^T dy(t) = \int_0^T \frac{dy}{dt} dt.$$

In this chapter, we will use the delta (Δ) notation when we are conducting Taylor series expansions to estimate changes in function. We will use the differential (d) notation for the case when we have dropped higher order terms.

Taylor Series and Two Independent Variables

Now, suppose that $y = y(x, t)$; that is, y is a function of the independent variables x and t. The Taylor series expansion can be generalized to two independent variables as follows:

$$\Delta y = \frac{\partial y}{\partial x} \cdot \Delta x + \frac{\partial y}{\partial t} \cdot \Delta t + \frac{1}{2} \cdot \frac{\partial^2 y}{\partial x^2} \cdot (\Delta x)^2 + \frac{1}{2} \cdot \frac{\partial^2 y}{\partial t^2} \cdot (\Delta t)^2 + \frac{1}{2} \cdot \frac{\partial^2 y}{\partial x \partial t} \cdot (\Delta x \Delta t) + ...$$

If $x = x(t)$ is a differentiable function of time t, then we have the approximation $\Delta x = x'(t)\Delta t$. Ignoring higher order terms in Δt, we obtain:

$$dy = \frac{\partial y}{\partial x} x'(t) dt + \frac{\partial y}{\partial t} dt \ ,$$

where we have expressed the result in differential form. This implies that

$$y(T) - y(0) = \int_0^T dy(t) = \int_0^T \left(\frac{\partial y}{\partial t} x'(t) + \frac{\partial y}{\partial t}\right) dt.$$

Note that for purposes of economy of notation we denoted $y(t) = y(x(t),t)$.

C. Itô's Lemma

Itô's Lemma is often regarded to be the Fundamental Theorem of Stochastic Calculus. Brownian motion processes are fractals that do not smooth as $\Delta t \to 0$. Newtonian calculus, which requires smoothing, cannot be used to differentiate or antidifferentiate Brownian motion functions. Hence, we will rely on Itô's Lemma to analyze continuous-time stochastic processes.

The Itô Process

First, we will consider an Itô process, which is a stochastic process X_t that can be expressed as follows:

$$dX_t = a(X_t, t)dt + b(X_t, t)dZ_t.$$

The drift of the process is a, while b^2 is the instantaneous variance and dZ_t is a standard Brownian motion process. Taking Δt to be a small change in time and expressing $a = a(X_t, t)$ and $b = b(X_t, t)$ in order to economize the notation, we can write:

$$\Delta X_t = a\Delta t + b\Delta Z_t,$$

where random variable $\Delta Z_t \sim N(0,\Delta t)$.

Now, suppose that $y = y(x, t)$ is an infinitely differentiable function with respect to the real variables x and t. Now, replace x with X_t (a stochastic process) so that $y = y(X_t, t)$. Thus, y itself becomes a stochastic process, since it is a function of a stochastic process X_t and time t. The Taylor series expansion above can be used to estimate Δy, the change in y, resulting from a change Δt in time:

$$\Delta y(X_t, t) = \frac{\partial y}{\partial t}\Delta t + \frac{\partial y}{\partial x}(a\Delta t + b\Delta Z_t) + \frac{1}{2}\cdot\frac{\partial^2 y}{\partial x^2}(a\Delta t + b\Delta Z_t)^2 + \frac{1}{2}\cdot\frac{\partial^2 y}{\partial t^2}(\Delta t)^2 + \frac{1}{2} \cdot \frac{\partial^2 y}{\partial x\,\partial t}(a\Delta t + b\Delta Z_t)\Delta t + \cdots$$

$$= \frac{\partial y}{\partial t}\Delta t + \frac{\partial y}{\partial x}(a\Delta t + b\Delta Z_t) + \frac{1}{2}\cdot\frac{\partial^2 y}{\partial x^2}[a^2(\Delta t)^2 + 2ab\Delta t\Delta Z_t + b^2(\Delta Z_t)^2] + \frac{1}{2}\cdot\frac{\partial^2 y}{\partial t^2}(\Delta t)^2 + \frac{1}{2}\cdot\frac{\partial^2 y}{\partial x\,\partial t}[a(\Delta t)^2 + b\Delta Z_t\Delta t] + \cdots$$

In the expansion above, we also economized the notation for the various partial derivatives evaluated at $(x,t)=(X_t,t)$ by denoting:

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial x}(X_t,t),\frac{\partial y}{\partial x}= \frac{\partial y}{\partial t}(X_t,t),\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial x^2}(X_t,t),\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y}{\partial t^2}(X_t,t),\frac{\partial^2 y}{\partial x\partial t}= \frac{\partial^2 y}{\partial x\partial t}(X_t,t).$$

When estimating Δy using the Taylor series expansion, all terms that are negligible compared to Δt as Δt approaches 0 can be dropped. Since $(Z_{t+\Delta t} - Z_t) \sim Z\sqrt{\Delta t}$ with $Z\sim$ N(0,1), the terms involving $(\Delta t)^2$, $\Delta t\Delta Z_t$, and higher order terms can all be dropped. Our expansion simplifies to:

(12) $$\Delta y = \frac{\partial y}{\partial t}\Delta t + \frac{\partial y}{\partial X_t}(a\Delta t + b\Delta Z_t) + \frac{b^2}{2}\cdot\frac{\partial^2 y}{\partial X_t^2}(\Delta Z_t)^2.$$

Since $(\Delta Z_t)^2 \sim Z^2\Delta t$, this term is not negligible. We now state a remarkable fact. We can actually replace $(\Delta Z_t)^2$ with Δt, where the random variable has seemingly disappeared. To show this, we will make use of the fact that Brownian motion increments are independent for disjoint intervals.

Demonstration that Δt can replace $(\Delta Z_t)^2$ in the differential Δy

We begin by subdividing the interval $[t, t + \Delta t]$ into n smaller subintervals $[t +(i-1)\Delta t/n, t + i\Delta t/n]$ for $i = 1, 2, \ldots n$. This means that the width of each subinterval equals $\Delta t/n$. In the Taylor series expansion above of Δy, the subintervals lead to $(\Delta Z_t)^2$ being replaced by $\sum_{i=1}^{n}(\Delta_i Z_t)^2$ with

$$\Delta_i Z_t = Z_{t+\frac{i\Delta t}{n}} - Z_{t+\frac{(i-1)\Delta t}{n}}.$$

Since

$$\Delta_i Z_t = Z_{t+\frac{i\Delta t}{n}} - Z_{t+\frac{(i-1)\Delta t}{n}} \sim N\left(0,\frac{\Delta t}{n}\right),$$

the variance of the random variable $\Delta_i Z_t$ is:

$$E[(\Delta_i Z_t)^2] = Var(\Delta_i Z_t) = \frac{\Delta t}{n}.$$

Notice that this result gives us the expected value of the random variable $(\Delta_i Z_t)^2$. Next we calculate the variance of $(\Delta_i Z_t)^2$. From the second equation above, we see that

$$\Delta_i Z_t \sim Z\sqrt{\frac{\Delta t}{n}},$$

with Z ~ N(0,1), which implies

$$(\Delta_i Z_t)^2 \sim Z^2\frac{\Delta t}{n}.$$

The solution to end-of-chapter exercise 10 verifies that Var $(Z^2c) = 2c^2$ for any $c > 0$. By the last equation above and the fact that $\text{Var}(Z^2c) = 2c^2$, we obtain:

$$Var[(\Delta_i Z_t)^2] = \frac{2(\Delta t)^2}{n^2}.$$

The Brownian motion increments $\Delta_i Z_t$ are independent random variables since the corresponding time intervals are disjoint. Since the variance of a sum of independent random variables is the sum of each of their variances, then:

$$Var\left(\sum_{i=1}^{n}(\Delta_i Z_t)^2\right) = \sum_{i=1}^{n} Var\ ((\Delta_i Z_t)^2) = \sum_{i=1}^{n}\frac{2(\Delta t)^2}{n^2} = \frac{2(\Delta t)^2}{n}.$$

Since we showed above that $E[(\Delta_i Z_t)^2] = \frac{\Delta t}{n}$, we see that

$$E\left[\sum_{i=1}^{n}(\Delta_i Z_t)^2\right] = \frac{\Delta t}{n} n = \Delta t.$$

Thus, the quantity $\sum_{i=1}^{n}(\Delta_i Z_t)^2$ has an expected value of Δt and a variance $2(\Delta t)^2/n$ that approaches 0 as n approaches infinity. Since $(\Delta Z_t)^2$ must approach $\sum_{i=1}^{n}(\Delta_i Z_t)^2$ as $n\to\infty$ and as Δt gets arbitrarily small, this implies that $(\Delta Z_t)^2$ has an expected value of Δt and a variance approaching $2(\Delta t)^2/n\to 0$. But a random variable with variance 0 is simply equal to its expected value. So, we conclude that $(\Delta Z_t)^2 = \Delta t$.

Itô's formula

Equation (12) above simplifies to:

$$\Delta y = \left(\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 y}{\partial x^2}\right)\Delta t + b\frac{\partial y}{\partial x}\Delta Z_t.$$

We can express this result which becomes *Itô's formula* in differential form:

$$dy(X_t, t) = \left(\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 y}{\partial x^2}\right)dt + b\frac{\partial y}{\partial x}dZ_t.$$

where the partial derivatives are evaluated at $(x,t) = (X_t,t)$.

Itô's Lemma

The differential that we obtained above is the stochastic calculus version of the differential from ordinary calculus. This result is known as *Itô's Lemma*:

> Given a real-valued function: $y = y(x,t)$, define the stochastic process $y(X_t,t)$, where X_t is an Itô process $dX_t = a(X_t,t)dt + b(X_t,t)dZ_t$ with Z_t denoting standard Brownian motion. By Itô's formula, the differential of the process $y(X_t,t)$ satisfies the equation:

$$dy(X_t, t) = \left[a\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 y}{\partial x^2}\right]dt + b\frac{\partial y}{\partial x}dZ_t$$

where the partial derivatives above are evaluated at $(x,t) = (X_t,t)$.

Applying Itô's Lemma

Evaluating stochastic integrals is generally trickier than evaluating ordinary real valued integrals. I recommend the following 3-step process to evaluate stochastic integrals:

1. As a first attempt, apply the form of the solution that mimics the solution for the analogous problem in ordinary calculus.
2. Invoke Itô's Lemma to find the differential of the attempted solution.
3. Integrate both sides of the differential and rearrange the result to solve for the desired stochastic integral.

Illustration: Applying Itô's Lemma

Suppose that we seek to evaluate $\int_0^T Z_t dZ_t$. Using the 3-step technique described above, we evaluate $\int_0^T Z_t dZ_t$ as follows:

1. Attempt the ordinary calculus solution which would suggest that $Y_T = \int_0^T Z_t dZ_t = \frac{1}{2}Z_T^2$.
2. Find the differential of our attempted solution $Y_T = \frac{1}{2}Z_T^2$ using Itô's Lemma. Choose $y(x,t) = \frac{1}{2}x^2$ so that $Y_t = y(Z_t,t) = \frac{1}{2}Z_t^2$. With $dX_t = dZ_t = 0{\cdot}dt + 1{\cdot}dZ_t$, and invoking Itô's Lemma, we have:

$$dY_t = dy(Z_t, t) = \left[0\cdot\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} + \frac{1}{2}\cdot 1^2\cdot\frac{\partial^2 y}{\partial x^2}\right]dt + 1\cdot\frac{\partial y}{\partial x}dZ_t$$
$$= \frac{\partial y}{\partial x}dZ_t + \frac{\partial y}{\partial t}dt + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}dt.$$

In Itô's Lemma, we must evaluate the partial derivatives at $(x,t) = (Z_t,t)$. Since $\frac{\partial y}{\partial x} = x|_{x=Z_t} = Z_t$, $\frac{\partial y}{\partial x} = 0$ and $\frac{\partial^2 y}{\partial x^2} = 1$, we have:

$$dY_t = Z_t dZ_t + \frac{1}{2}dt.$$

3. We will integrate both sides of this equation for $0 \le t \le T$. The left side of the following equation is based on our discussion concerning the integral of a stochastic differential at the start of Section 6.1.3. The right side of the following is based on the equation immediately above:

$$Y_T - Y_0 = \int_0^T dY_t = \int_0^T Z_t dZ_t + \int_0^T \frac{1}{2}dt$$

$$\frac{1}{2}(Z_T)^2 - \frac{1}{2}(Z_0)^2 = \int_0^T Z_t dZ_t + \int_0^T \frac{1}{2} dt.$$

$$\frac{1}{2}(Z_T)^2 = \int_0^T Z_t dZ_t + \frac{1}{2}T.$$

Solving for the desired integral results in

$$\int_0^T Z_t dZ_t = \frac{1}{2}(Z_T)^2 - \frac{1}{2}T.$$

Note that the result is a stochastic process. Also, observe that

$$E\left[\frac{1}{2}(Z_T)^2 - \frac{1}{2}T\right] = \frac{1}{2}Var[Z_T] - \frac{1}{2}T = 0$$

which gives the correct expectation, and clearly, $Var[Z_T] = T$.

Intuitively, the reason why the Brownian motion integral and ordinary calculus integral result in different forms of solutions is because $x(t)$ is a continuously differentiable function while Z_t is not differentiable. More precisely, $dx(t) = x'(t)dt$ and $(dx(t))^2 = (x'(t))^2(dt)^2$, while dZ_t has standard deviation equal to $\sqrt{dt}$ and $(dZ_t)^2 = dt$. This means that the second-order term in the Taylor expansion of $d\left[\frac{1}{2}Z_t^2\right]$ becomes important since $(dZ_t)^2 = dt$. In ordinary calculus, only the first-order term in the Taylor expansion of $d\left[\frac{1}{2}\left(x(t)\right)^2\right]$ is important since $(dx(t))^2 = (x'(t))^2(dt)^2$ and $(dt)^2$ is negligible insofar as integration is concerned.

Application: Geometric Brownian Motion

Geometric Brownian motion is an essential model for characterizing the stochastic process for a security with value S_t at time t:

(13) $$dS_t = \mu S_t dt + \sigma S_t dZ_t.$$

Recall that μ and σ are the geometric mean security return and the standard deviation of the security return per unit of time. We wish to determine the value of the security S_t as a function of time. First, rewrite the differential above in the form:

(14) $$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t.$$

The reason why this model is used extensively to model security prices S_t is that the instantaneous return on the security over the time interval $[t,t+dt]$ is the known drift μdt plus a random Brownian motion compoment σdZ_t. The drift is related to the time value of money and the Brownian motion term reflects the unknowable random factors affecting the security's price. Since Brownian motion has a normal distribution, we are, of course, assuming a particular type of random fluctuation. It is a subject of much debate in finance as to the merits of this choice to model securities.

Next, we will integrate both sides of equation (14) from 0 to *T*. To evaluate the integral of the left side, use our 3 step procedure involving Itô's Lemma. Step 1 is to evaluate the integral as though S_t is a real-valued function:

$$\int_0^T \frac{dS_t}{S_t} = \ln(S_t)\Big|_0^T = \ln(S_T) - \ln(S_0).$$

Since lnS_0 is a constant, we can ignore it for purposes of computing the differential of the right side function $lnS_t - lnS_0$. Step 2 is to take the differential of lnS_t using Itô's Lemma applying it to the function $y = y(S_t,t) = lnS_t$. In this formulation, dy from Itô's formula is $d(lnS_t)$, a is μS_t and b is σS_t. We substitute these values into Itô's formula as follows:

$$d(lnS_t) = \left[\mu S_t \frac{\partial y}{\partial S} + \frac{\partial y}{\partial t} + \frac{1}{2}\sigma^2 S_t^{\,2} \frac{\partial^2 y}{\partial S^2}\right]dt + \sigma S_t \frac{\partial y}{\partial S} dZ_t$$

Following standard rules for differentiation, the above equation is written:

$$d(\ln S_t) = \left[\mu S_t \frac{1}{S_t} + 0 - \frac{1}{2}\sigma^2 S_t^{\,2} \frac{1}{S_t^{\,2}}\right]dt + \sigma S_t \frac{1}{S_t} dZ_t ,$$

which simplifies to:

$$d\ln(S_t) = \left[\mu - \frac{1}{2}\sigma^2\right]dt + \sigma dZ_t$$

This equation would be the basis of a Monte Carlo simulation of geometric Brownian motion for stock price behavior should we wish to run one. Now, perform Step 3 by integrating both sides of this equation from $t = 0$ to $t = T$, which results in:

$$\int_0^T d\ln(S_t) = \int_0^T \left[\left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t\right]$$

$$\ln(S_t)\Big|_0^T = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\Big|_0^T$$

$$lnS_T - lnS_0 = ln\frac{S_T}{S_0} = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma Z_T.$$

Exponentiating both sides of this equation and multiplying by S_0 yields:

(15) $$S_T = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma Z_T}.$$

Notice that in the exponent of *e*, there is an extra factor of $-\sigma^2/2$ that would not appear in the solution to the classic problem $dS_t = \mu dt$. As we discussed in Chapter 4, the classic problem has the solution $S_T = S_0 e^{\mu T}$. Itô's formula shows that the effect of the Brownian motion term results in this increment to drift equal to $-\sigma^2/2$ in the solution for S_T. Also recall that the above solution was used to obtain the heuristic probabilistic derivation of the price of a European call option (Chapter 5), except that μ was replaced with the return *r* on a riskless bond. Next, we calculate the expected value of the logarithmic return:

(16) $$E\left[ln\frac{S_T}{S_0}\right] = E\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma Z_T\right] = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma E[Z_T] = \left(\mu - \frac{1}{2}\sigma^2\right)T = \alpha T.$$

Returns and Price Relatives

The constant $\alpha = \mu - \sigma^2/2$ is known as the *mean logarithmic return* of the security per unit time. If we express the security price in the form:

(17) $$S_T = S_0 e^{\alpha T + \sigma Z_T},$$

then $ln(S_T/S_0) = \alpha T + \sigma Z_T$ is known as the *log of price relative* or the *logarithmic return*. It is also useful to calculate the variance of the logarithmic return.

(18) $$Var\left(ln\frac{S_T}{S_0}\right) = E\left[\left(ln\frac{S_T}{S_0} - \alpha T\right)^2\right] = E[\sigma^2 Z_T^2] = \sigma^2 E[Z_T^2] = \sigma^2 T.$$

In comparison, let's examine the expected value and variance of the arithmetic return on S_T over time *T*. The *arithmetic return over time T* is defined as $r = S_T/S_0 - 1$. To derive expected arithmetic return over time *T*, we first observe that $E[\alpha T + \sigma Z_T] = \alpha T$ and $Var[\alpha T + \sigma Z_T] = E[\sigma^2 (Z_T)^2] = \sigma^2 T$. Since $\alpha T + \sigma Z_T \sim N(\alpha T, \sigma^2 T)$, we have:

(19) $$E[r] = E[e^{\alpha T + \sigma Z_T} - 1] = e^{\alpha T + \frac{1}{2}\sigma^2 T} - 1.$$

We leave it as end-of-chapter Exercise 18 to verify that the variance of the arithmetic return is:

(20) $$Var[r] = \left(e^{\sigma^2 T} - 1\right)e^{2\alpha T + \sigma^2 T}.$$

Because of the exponential nature of the arithmetic return in contrast to the linear nature of the logarithmic return, the expected value and variance of the arithmetic return grows exponentially with time *T*, while the expected value and variance of the logarithmic return grows linearly with time *T*.

Itô's Formula: Numerical Illustration

Suppose, for example, that the following Itô process describes the price path S_t of a given stock:

$$dS_t = .01S_t dt + .015S_t dZ_t$$

The differential for this process describes the infinitesimal change in the price of the stock. The expected rate of return per unit time is .01 and the standard deviation of the return per unit time is .015. The solution for this equation giving the actual price level at a point in time is given by:

$$S_T = S_0 e^{\left[\left(.01 - \frac{.015^2}{2}\right)T + .015Z_t\right]}$$

Suppose that one needed a return (or, log of price relative) and variance over time *T* for the stock. The expected value and variance of the log of price relative are given by:

$$E\left[\ln\frac{S_T}{S_0}\right] = \alpha T = \mu T - \frac{1}{2}\sigma^2 T = \left(.01 - \frac{.015^2}{2}\right)\cdot 1 = .0098875$$

$$Var\left[\ln\frac{S_T}{S_0}\right] = \sigma^2 T = .015^2 \cdot 1 = .000225$$

The expected value and variance of the arithmetic return over a single period are given by:

$$E[r] = e^{\alpha T + \frac{1}{2}\sigma^2 T} - 1 = e^{\left(.0098875\times 1 + \frac{1}{2}.015^2 \times 1\right)} - 1 = .01005$$

and

$$Var[r] = \left(e^{\sigma^2 T} - 1\right)e^{2\alpha T + \sigma^2 T} = \left(e^{.015^2\times 1} - 1\right)e^{\left(2\times .0098875\times 1 + .015^2 \times 1\right)} = .00022957.$$

Figure 1 depicts a simulated geometric Wiener process over length of time 10,000, with $S_0 = 100$, $\mu = .00001$ and $\sigma = .001$. This diagram was obtained from a simulation of a random process with 10,000 data points.

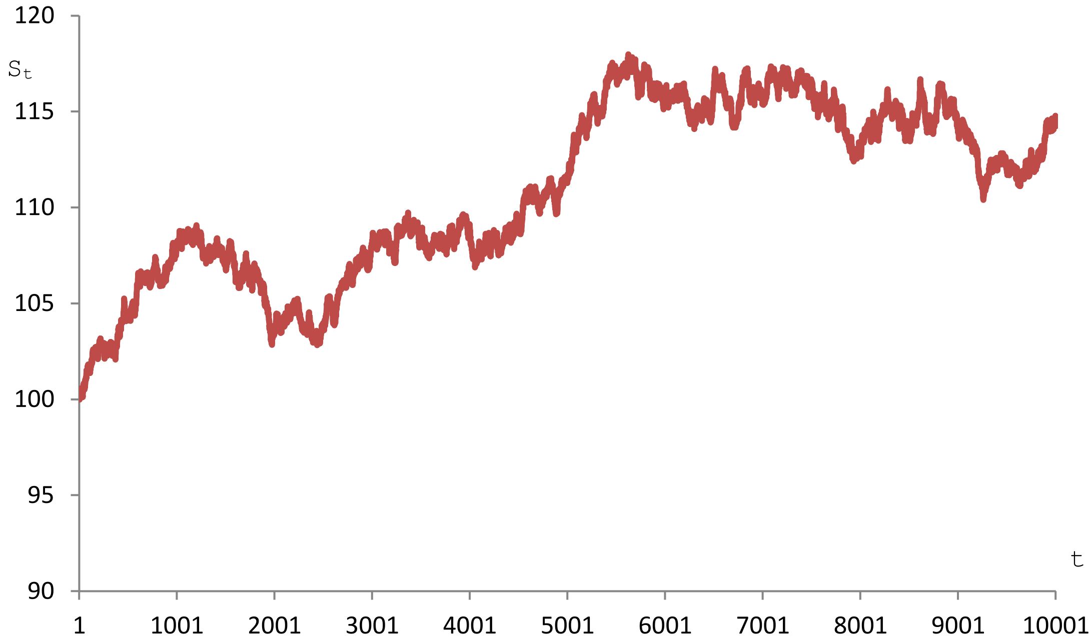

Figure 1: Geometric Brownian Motion: S_0 = 100, μ = .00001, σ = .001, n = 10,000

References

Knopf, Peter M. and John L. Teall (2015): *Risk Neutral Pricing and Financial Mathematics: A Primer*, Waltham, Massachusetts: Elsevier, Inc.

Exercises

1. Find the differential of the stochastic process $X_t = t^2 Z_t$.

2. Find the differential of the stochastic process $X_t = .06t + 100 + .02Z_t$.

3. Consider the following functions of a real-valued variable and of a stochastic process. Evaluate each and then contrast them:

   a. $\int_0^t 5dX_s$ if $X_t = t^2$.

   b. $\int_0^t 5dZ_s$ if Z_s is Brownian motion.

4. Suppose that X_t and Y_t are stochastic processes and C and D are real numbers. Verify that

$$\int_a^b (CdX_t + DdY_t) = C(X_b - X_a) + D(Y_b - Y_a).$$

5. Suppose that $S_{t-1} = 0$ and that potential outcomes S_t for a one time-period submartingale process are +1 with probability .6 and -1 with probability .4.

   a. Compute the expected value $E\left[S_t | S_{t-1}\right]$ of this process.

   b. Is the process in this problem a martingale?

   c. Compute the drift of the process.

6. Consider a one time period, two potential outcome framework where there exists Company Q stock currently selling for \$50 per share and a riskless \$100 face value T-Bill currently selling for \$80. Suppose Company Q faces uncertainty, in that it will pay its owner either \$30 or \$70 in one year. Further assume that the physical probability that the stock price will drop is .2.

   a. List the risk neutral probabilities for this payoff space.

   b. Value call and put options on this stock, with exercise prices equal to X = \$60.

   c. Does put-call parity hold for this example?

7. Rollins Company stock currently sells for \$12 per share and is expected to be worth either \$10 or \$16 in one year. The current riskless return rate is .125 and the physical probability that the stock price will increase is .75. List the risk neutral probabilities for this payoff space.

8. Suppose that we pay $S_0 = .7$ to purchase a security, with potential payoffs given as follows: $S_1 = 2$, $S_2 = 1$ and $S_3 = 0$ such that under physical probabilities, $E_{\mathbb{P}}[S] = .8$ and a variance equals $E_{\mathbb{P}}\left[S - E_{\mathbb{P}}[S]\right]^2 = .76$. Find the risk-neutral probabilities in measure $\mathbb{Q}$ based on the market price of the stock.

9. Suppose that we have a random variable $X \sim N(\mu, 1)$.

   a. Write the density function $f_{\mathbb{P}}(x)$ under probability measure $\mathbb{P}$.

   b. Suppose that we change the probability measure of X to measure $\mathbb{Q}$, which is an equivalent martingale measure to $\mathbb{P}$. Write the density function $f_{\mathbb{Q}}(x)$ under probability measure $\mathbb{Q}$.

10. In the text, we made the claim that the variance of the random variable Z^2c where $Z \sim N(0,1)$ and $c > 0$ equals $2c^2$.

a. First, show that the variance of the random variable Z^2 where $Z \sim N(0,1)$ equals 2.

b. Use the result of part a to show that the variance of the random variable Z^2c where $Z \sim N(0,1)$ and $c > 0$ equals $2c^2$.

11. Suppose a stock price S_t evolves according to $dS_t = tdt + \sigma dZ_t$ where Z_t is standard Brownian motion, $\sigma > 0$ is a constant, and S_0 is the initial price of the stock. Derive an equation to find the price of the stock at time T.

12. Suppose that the logarithmic return α on a stock follows a Wiener process (Brownian motion process) with drift, an expected value over one year equal to 5% and a variance equal to .09; that is, $\alpha \sim N(\mu,\sigma^2)$ with $\mu = .05$ and $\sigma^2 = .09$. Find the expected value and variance of the arithmetic return for the stock.

13. Suppose that the log of price relatives (instantaneous returns) for a stock follows a Wiener process with drift, an expected value equal to 6% per annum and a variance equal to .08 per annum. What are the expected value and variance of the arithmetic return r for the stock over one year?

14. Suppose that the following Itô process describes the price of a given stock after t weeks:

$$dS_t = .001S_t dt + .02S_t dZ_t$$

a. What is the solution to this stochastic differential equation?
b. Suppose that there are 52 periods in a year. What are the expected value and variance of the log of price relative for this stock over a 52-week period?

15. Suppose that X_t is a geometric Brownian motion process with $dX_t = \mu X_t dt + \sigma X_t dZ_t$. Consider a function Y_t of X_t with $Y_t = (X_t)^n$. Derive an expression for dY_t.

16. Suppose that a particular derivative instrument with price S_t satisfies the differential $dS_t = \mu(M - S_t)dt + \sigma(M - S_t)dZ_t$ and initial value S_0 with $0 < S_0 < M$. Find the solution for S_t that is valid as long as $0 < S_t < M$.

17. Suppose that a stock price S_t satisfies the model $dS_t = \mu dt + Z_t dZ_t$ where Z_t is standard Brownian motion, $\mu > 0$ is a constant, and S_0 is the initial price of the stock. Derive an expression to price the stock at time t.

18. Derive the variance formula given by equation (20) in Section C for the arithmetic return of a security that follows a geometric Brownian motion of the form $S_T = S_0 e^{\alpha T + \sigma Z_T}$.

Solutions

1. By the product rule, $dX_t = t^2dZ_t + Z_t d(t^2) = t^2dZ_t + 2tZ_t dt$.

2. By linearity: $dX_t = .06dt + .02dZ_t$.

3.a. $\int_0^t 5dX_s = 5X_s|_0^t = 5s^2|_0^t = 5(t)^2 - 5(0)^2 = 5t^2$.

b. $\int_0^t 5dZ_s = 5Z_s|_0^t = 5Z_t - 5Z_0 = 5Z_t$.

The solution for part a is a real-valued function of time. The solution for part b is the stochastic process 5 times standard Brownian motion.

4. Using the linearity property followed by the integral of a stochastic differential property in Section 6.1.3, we obtain:

$$\int_a^b (CdX_t + DdY_t) = C\int_a^b dX_t + D\int_a^b dY_t = C(X_b - X_a) + D(Y_b - Y_a).$$

5.a. The expected value of the process given S_{t-1} is computed as follows:

$$E\left[S_t|S_{t-1}\right] = \sum_i S_{t,i}p_i = (1+0)(.6) + (-1+0)(.4) = 0.2$$

b. No – since $E[S_t|S_{t-1}]=.2 > 0=S_{t-1}$, this process is a submartingale.

c. The drift of this process is $E[S_t|S_{t-1}]=.2$.

6.a. Risk-neutral probabilities are computed as follows:

$$\begin{bmatrix} 30 & 70 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} \psi_{0,1,1} \\ \psi_{0,1,2} \end{bmatrix} = \begin{bmatrix} 50 \\ 80 \end{bmatrix} \; ; \quad \begin{matrix} \psi_{0,1,1} = .15 \\ \psi_{0,1,2} = .65 \end{matrix}$$

$$q_{0,1,1}/(\psi_{0,1,1} + \psi_{0,1,2}) = .1875$$

$$q_{0,1,2}/(\psi_{0,1,1} + \psi_{0,1,2}) = .8125$$

b. $c_0 = (.8 \times .1875 \times 0) + (.8 \times .8125 \times \$10) = 0 + \$6.5 = \$6.5$

$p_0 = (.8 \times .1875 \times \$30) + (.8 \times .8125 \times 0) = \$4.5 + 0 = \$4.5$

c. Yes: $\$6.5 + (.8 \times \$60) = \$4.5 + \$50$

7. Risk-neutral probabilities are computed as follows:

$$\begin{bmatrix} 10 & 16 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} \psi_{0,1,1} \\ \psi_{0,1,2} \end{bmatrix} = \begin{bmatrix} 12 \\ 88.888889 \end{bmatrix} \; ; \quad \begin{matrix} \psi_{0,1,1} = .37037 \\ \psi_{0,1,2} = .51852 \end{matrix}$$

$$q_{0,1,1}/(\psi_{0,1,1} + \psi_{0,1,2}) = .4166667$$

$$q_{0,1,2}/(\psi_{0,1,1} + \psi_{0,1,2}) = .5833333$$

8. Consider the following statements, which state that the risk-neutral probabilities need to sum to one, the expected value of the $\mathbb{Q}$ distribution must equal $S_0 = .7$ ($\mathbb{Q}$ is an equivalent martingale measure to $\mathbb{P}$) and the variance of both must be equal .76:

$$\sum_i q_i = q_1 + q_2 + q_3 = 1$$

$$\sum_i S_i q_i \;=\; 2q_1 + 1q_2 \;+ 0q_3 = E_{\mathbb{Q}}[S] = \;.7$$

$$E_{\mathbb{Q}}\left[S - E_{\mathbb{Q}}[S]\right]^2 = \sum_i \left[S_i - E_{\mathbb{Q}}[S]\right]^2 q_i \;=\; (2-.7)^2 \times q_1 + (1-.7)^2 \times q_2 + (0-.7)^2 \times q_3 \;=\; .76$$

We solve this system simultaneously as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1.69 & .09 & .49 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \\ .76 \end{bmatrix}$$

$$\begin{bmatrix} -.245 & .2 & .5 \\ .49 & .6 & -1 \\ .755 & -.8 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ .7 \\ .76 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} .275 \\ .15 \\ .575 \end{bmatrix}$$

Similarly, one finds the physical probabilities by solving the system:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1.44 & .04 & .64 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ .8 \\ .76 \end{bmatrix},$$

and one obtains: $p_1=.3$, $p_2=.2$, and $p_3=.5$.

9. a. $f_{\mathbb{P}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$

   b. $f_{\mathbb{Q}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

10. a. First note that $E[Z^2] = Var[Z] = 1$. Thus:

$$Var(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z^2-1)^2 \, e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z^4 - 2z^2 + 1) \, e^{-z^2/2} dz$$

We already know that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1 \; and \; Var(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \, e^{-z^2/2} dz = 1.$$

So there is left to evaluate

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 \, e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^3 \, e^{-z^2/2} z dz.$$

Choose $u = z^3$ and $dv = e^{-z^2/2} z dz$ and integrate by parts. Since $du = 3z^2 dz$ and $v = -e^{-\frac{z^2}{2}}$, then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^3 \, e^{-z^2/2} z dz = -\frac{1}{\sqrt{2\pi}} z^3 e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} + \frac{3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \, e^{-\frac{z^2}{2}} dz = 3Var(Z) = 3$$

This gives

$$Var(Z^2) = 3 - 2 + 1 = 2.$$

b. First note that $E[Z^2c]=c$. It is now straightforward to calculate $Var[Z^2c]$:

$Var[Z^2c]=E[(Z^2c - c)^2]=c^2E[(Z^2-1)^2]=c^2Var[Z^2]=2c^2$.

11. We solve $dS_t = tdt + \sigma dZ_t$ as follows:

$$\int_0^T dS_t = \int_0^T tdt + \sigma \int_0^T dZ_t$$

$$S_T - S_0 = \frac{1}{2}t^2 \,|_0^T + \sigma Z_t|_0^T$$

$$S_T - S_0 = \frac{1}{2}T^2 - \frac{1}{2}0^2 + \sigma Z_T - \sigma \times 0.$$

$$S_T - S_0 = \frac{1}{2}T^2 + \sigma Z_T.$$

$$S_T = S_0 + \frac{1}{2}T^2 + \sigma Z_T.$$

12. The mean and variance of arithmetic returns are computed as follows:

$$E[r] = E\left[\frac{S_T}{S_o} - 1\right] = e^{\alpha T + \frac{1}{2}\sigma^2 T} - 1 = e^{.05+.045} - 1 = .09966$$

$$Var[r] = \left(e^{\sigma^2 T} - 1\right)e^{2\alpha T + \sigma^2 T} = \left(e^{.09} - 1\right)e^{2\times .05+.09} = .11388$$

13. The expected value and variance are computed as follows:

$$E[r] = E\left[\frac{S_T}{S_o} - 1\right] = e^{\alpha T + \frac{1}{2}\sigma^2 T} - 1 = e^{.06+.04} - 1 = .10517$$

$$Var[r] = \left(e^{\sigma^2 T} - 1\right)e^{2\alpha T + \sigma^2 T} = \left(e^{.08} - 1\right)e^{.2} = .1017$$

14. a.

$$S_T = S_0 e^{\left[\left(.001 - \frac{.02^2}{2}\right)T + .02 Z_T\right]} = S_0 e^{(.0008T + .02Z_T)}$$

b.

$$E\left[\ln\frac{S_{52}}{S_0}\right] = \left[\left(.001 - \frac{.02^2}{2}\right).52\right] = .0416$$

$$Var\left[\ln\frac{S_{52}}{S_0}\right] = \sigma^2 T = .02^2 \cdot 52 = .0208$$

15. Define $y(x,t) = x^n$ so that $Y_t = y(X_t,t)$. The partial derivatives of y evaluated at $(x,t) = (X_t,t)$ are as follows:

$$\frac{\partial y}{\partial x} = nX_t^{n-1};\ \frac{\partial^2 y}{\partial x^2} = (n^2 - n)X_t^{n-2};\ \frac{\partial y}{\partial t} = 0.$$

With $a = \mu X_t$ and $b = \sigma X_t$, we apply Itô's Lemma and obtain:

$$dY_t = \left[\mu X_t n X_t^{n-1} + \tfrac{1}{2}\sigma^2 X_t^2 (n^2 - n)X_t^{n-2}\right]dt + \sigma X_t n X_t^{n-1} dZ_t$$

$$= \left[n\mu + \frac{1}{2}(n^2 - n)\sigma^2\right]X_t^n dt + n\sigma X_t^n dZ_t.$$

16. First, divide both sides of the differential by $M – S_t$ to obtain:

$$\frac{dS_t}{M - S_t} = \mu dt + \sigma dZ_t.$$

Since the integral of $dS_t / (M – S_t)$ for real-valued functions S_t equals $–ln(M – S_t)$, we will use the expression $ln(M – S_t)$ to obtain the correct solution for the stochastic process S_t. Recall that Itô's formula is:

$$dy = [a\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 y}{\partial x^2}]dt + b\frac{\partial y}{\partial x}dZ_t$$

Define $y(x,t)=ln(M\text{-}x)$ so that $Y_t=y(S_t,t)$. The partial derivatives of y evaluated at $(x,t) = (S_t,t)$ are as follows:

$$\frac{\partial y}{\partial x} = \frac{-1}{M - S_t};\ \frac{\partial^2 y}{\partial x^2} = \frac{-1}{(M - S_t)^2};\ \ \frac{\partial y}{\partial t} = 0$$

We now use Itô's Lemma, where $a = \mu(M - S_t)$ and $b = \sigma(M - S_t)$:

$$d(\ln(M - S_t)) = \mu(M - S_t)\left(\frac{-1}{M - S_t}\right)dt + 0 + \frac{1}{2}\sigma^2(M - S_t)^2\left(\frac{-1}{(M - S_t)^2}\right)dt$$
$$+\sigma(M - S_t)\left(\frac{-1}{M - S_t}\right)dZ_t$$
$$= -\mu dt - \tfrac{1}{2}\sigma^2 dt - \sigma dZ_t = -\left(\mu + \tfrac{\sigma^2}{2}\right)dt - \sigma dZ_t.$$

Changing the variable from t to s and integrating from 0 to t results in

$$\ln(M - S_t) - \ln(M - S_0) = -\left(\mu + \frac{\sigma^2}{2}\right)t - \sigma Z_t$$

or

$$\ln\left(\frac{M - S_t}{M - S_0}\right) = -\left(\mu + \frac{\sigma^2}{2}\right)t - \sigma Z_t.$$

Exponentiating we have:

$$\frac{M - S_t}{M - S_0} = e^{-\left(\mu + \frac{\sigma^2}{2}\right)t - \sigma Z_t}.$$

Solving for S_t gives:

$$S_t = M - (M - S_0)e^{-\left(\mu + \frac{\sigma^2}{2}\right)t - \sigma Z_t}.$$

17. Employ the 3-step technique for solving stochastic differential equations as follows:

1. Attempt the ordinary calculus solution: $\int_0^T(\mu dt + Z_t dZ_t) = \mu T + \frac{1}{2}Z_T^2$.
2. Find the differential of $\mu T + \frac{1}{2}Z_T^2$ using Itô's Lemma. First, define the function

$F(x,t) = \mu t + \frac{1}{2}x^2$, so that $F(Z_t,t) = \mu t + \frac{1}{2}Z_t^2$. Invoking Itô's Lemma with $a = 0$ and $b = 1$, we have:

$$dF(Z_t,t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)dt + \frac{\partial F}{\partial x}dZ_t = \left(\mu + \frac{1}{2}\right)dt + Z_t dZ_t = dS_t + \frac{1}{2}dt.$$

3. Integrating both sides of this equation yields:

$$\int_0^T dF(Z_t,t) = \int_0^T dS_t + \frac{1}{2}\int_0^T dt$$

$$\mu T + \frac{1}{2} Z_T^2 = S_T - S_0 + \frac{1}{2} T$$

Solving for S_T results in $S_T = S_0 + \left(\mu - \frac{1}{2}\right) T + \frac{1}{2} Z_T^2$.

18. We already derived the expected value of the arithmetic return $r = S_T/S_0 - 1$ given by equation (19) in Section 6.5.5: $E[r] = e^{\alpha T + \frac{1}{2}\sigma^2 T}$. The variance of r is then:

$$Var[r] = E\left[\left((e^{\alpha T + \sigma Z_T} - 1) - \left(e^{\alpha T + \frac{1}{2}\sigma^2 T} - 1\right)\right)^2\right] = E\left[\left(e^{\alpha T + \sigma Z_T} - e^{\alpha T + \frac{1}{2}\sigma^2 T}\right)^2\right]$$

$$= e^{2\alpha T} E\left[\left(e^{\sigma Z_T} - e^{\frac{1}{2}\sigma^2 T}\right)^2\right] = e^{2\alpha T} E\left[e^{2\sigma Z_T} - 2e^{\sigma Z_T + \frac{1}{2}\sigma^2 T} + e^{\sigma^2 T}\right].$$

Since $2\sigma Z_T \sim N(0, 4\sigma^2 T)$ and $\sigma Z_T + \sigma^2 T/2 \sim N(\sigma^2 T/2, \sigma^2 T)$, using equation (26) in Section 2.6.4, we obtain:

$$E[e^{2\sigma Z_T}] = e^{2\sigma^2 T} \ and\ E\left[e^{\sigma Z_T + \frac{1}{2}\sigma^2 T}\right] = e^{\sigma^2 T}.$$

Substituting these results in the calculation of $Var[r]$, we see that:

$$Var[r] = e^{2\alpha T}\left(e^{2\sigma^2 T} - 2e^{\sigma^2 T} + e^{\sigma^2 T}\right) = e^{2\alpha T + \sigma^2 T}\left(e^{\sigma^2 T} - 1\right).$$