Chapter 5 Portfolios, Efficiency and the Capital Asset Pricing Model

The objectives of this chapter are to enable you to:

- *Understand the process of combining of securities into portfolios*
- *Understand measurement of portfolio return and risk*
- *Appreciate the importance of diversification*
- *Understand how diversification is related to security returns covariance and portfolio size*
- *Consider the impact of internationalization on portfolio risk and efficiency*
- *Combine riskless and risky securities to form an efficient portfolio*
- *Understand distinctions between market and firm-specific risk*
- *Understand the relationship between return and risk*
- *Compute beta and apply it to risk-adjusted discount rates for present value analysis*

5.A. INTRODUCTION

In Chapter 4, we learned about assessing the return and risk of a single security or investment. In this chapter, we will learn how to do the same for a portfolio. A portfolio is simply a collection of investments. The entire set of an investor's holdings is considered to be his portfolio. It may be reasonable for an investor to be concerned with the performance of individual securities only to the extent that their performance affects the performance of his overall portfolio of investments. Thus, the performance of the portfolio is of primary importance to the investor. The return of an investor's portfolio is simply a weighted average of the returns of the individual securities that comprise his portfolio. The expected return of a portfolio may be calculated either as a function of potential portfolio returns and their associated probabilities (a weighted average of potential returns as in Chapter 4) or as a simple weighted average of the expected individual security returns. However, the risk of the portfolio is somewhat more complicated to determine. Generally, the portfolio variance or standard deviation of returns will be less than a weighted average of the individual security variances or standard deviations. This reduction on portfolio risk will be intensified as the portfolio becomes more diversified; that is, portfolio risk is reduced when the selected securities are more dissimilar and when the number of securities in the portfolio increases.

5.B. PORTFOLIO RETURN

The expected return of a portfolio can be calculated using Equation (12.10) where the subscript (p) designates the portfolio and the subscript (j) designates one particular outcome out of (m) potential outcomes:

(4.10) $$E[R_p] = \sum_{j=1}^{m} R_{p \cdot j} \cdot P_j \quad .$$

Thus, the expected return of a portfolio is simply a weighted average of the potential portfolio returns where the outcome probabilities serve as the weights. This is just how we computed the expected return of a security in Chapter 4.

For many portfolio management applications, it is useful to express portfolio return as a function of the returns of the individual securities that comprise the portfolio. This is often because we want to know how a particular security will affect the return and risk of our overall holdings or portfolio. For example, consider a portfolio made up of two securities, one and two. The expected return of security one is 10% and the expected return of security two is 20%. If forty percent of the dollar value of the portfolio is invested in security one (that is, [w_1] = .40), and the remainder is invested in security two ([w_2] = .60), the expected return of the portfolio may be determined by Equation (5.1):

(5.1) $$E[R_p] = \sum_{i=1}^{n} w_i \cdot E[R_i]$$

$$E[R_p] = (.4 \cdot .10) + (.6 \cdot .20) = .16 \quad .$$

The subscript (i) designates a particular security, and weights [w_i] are the portfolio proportions. That is, a security weight (w_i) specifies how much money is invested in Security (i) relative to the total amount invested in the entire portfolio. For example, [w_1] is:

$$w_1 = \frac{\text{\$ invested in security 1}}{\text{Total \$ invested in the portfolio}}$$

Thus, portfolio return is simply a weighted average of individual security returns.

5.C. PORTFOLIO VARIANCE

Because risky securities often behave quite differently, the variance of portfolio returns is not simply a weighted average of individual security variances. In fact, in this section, we will demonstrate that combining securities into portfolios may actually result in risk levels lower than those of any of the securities comprising the portfolio. That is, in some instances, we can combine a series of highly risky assets into a relatively safe portfolio. The risk of this portfolio in terms of variance of returns can be determined by solving the following double summation:

(5.2) $$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$$

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Consider the portfolio constructed in section 5.B. If the standard deviation of returns on securities one and two were .20 and .30, respectively, and the correlation coefficient (ρ_{ij}) between returns on the two securities were .5, the resultant standard deviation of the portfolio would be .23, the square root of its .0504 variance level:

(5.3) $$\sigma_p^2 = (.4 \cdot .4 \cdot .2 \cdot .2 \cdot .1) + (.4 \cdot .6 \cdot .2 \cdot .3 \cdot .5) + (.6 \cdot .4 \cdot .3 \cdot .2 \cdot .5) + (.6 \cdot .6 \cdot .3 \cdot .3 \cdot .1) = .23\,;$$
$$\sigma_p^2 = .0504$$

More generally,

(5.4) $$\sigma_p^2 = (w_1 \cdot w_1 \cdot \sigma_1 \cdot \sigma_1 \cdot \rho_{11}) + (w_1 \cdot w_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho_{12}) + (w_2 \cdot w_1 \cdot \sigma_2 \cdot \sigma_1 \cdot \rho_{21}) + (w_2 \cdot w_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \rho_{22})$$

Notice that both counters (i) and (j) are set equal to one to begin the double summation process. Thus, in the first set of parentheses of Equations (5.3) and (5.4), since both (i) and (j) equal one, both portfolio weights (w_i) and (w_j) equal .4; (σ_i) and (σ_j) equal .2. The coefficient of correlation between any variable and itself must be one; therefore, (ρ_{11}) equals one. After variables are substituted into Equation (5.2) for (i) equals 1 and (j) equals 1, the counter of the inside summation is increased to two. Thus, in the second set of parentheses, (i) equals 1 and (j) equals 2. Hence, (w_i) equals .4, (w_j) equals .6, (σ_i) equals .2, and (σ_j) equals .3. Since the number of securities comprising the portfolio (n) is two, the inside summation is completed. We now increase the counter of the outside summation (i) to two and begin the inside summation over again (by setting j equal to 1). Thus, in the third set of parentheses, (i) equals two and (j) equals one. The correlation coefficient (ρ_{21}) must equal .5 because it must be identical to (ρ_{12}). We now increase the counter of the inside summation to two; in the fourth set of parentheses both (i) and (j) equal two. Since both counters now equal (n), (5.3) can be simplified and solved. It is important to realize that (i) and (j) are merely counters; they do not necessarily refer to any specific security consistently throughout the summation process. By simplifying the expressions in the first and fourth sets of parentheses, and combining the terms in the second and third sets, one can simplify Equation (5.3):

(5.5) $$\sigma_p^2 = (.4^2 \cdot .2^2) + (.6^2 \cdot .3^2) + 2(.4 \cdot .6 \cdot .2 \cdot .3 \cdot .5) = .0504$$

Therefore, when a portfolio is comprised of two securities, its variance can be determined by Equation (5.6):

(5.6) $$\sigma_p^2 = (w_1^2 \cdot \sigma_1^2) + (w_2^2 \cdot \sigma_2^2) + 2(w_1 \cdot w_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho_{12})$$

Equation (5.6) allows us to determine portfolio variance without having to work through the double summation only when (n) equals two. Larger portfolios require the use of some form of Equation (5.2). However, the number of sets of parentheses to work through and then add is equal to the number of securities in the portfolio squared (n^2). Equation (5.2) can be simplified to an equation with a form similar to that of (5.6). For example, if the portfolio were to be comprised of three securities, Equation (5.6) would change to:

(5.6.a) $$\sigma_p^2 = (w_1^2 \cdot \sigma_1^2) + (w_2^2 \cdot \sigma_2^2)(w_3^2 \cdot \sigma_3^2) + 2(w_1 \cdot w_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho_{12}) + 2(w_1 \cdot w_3 \cdot \sigma_1 \cdot \sigma_3 \cdot \rho_{13}) + 2(w_2 \cdot w_3 \cdot \sigma_2 \cdot \sigma_3 \cdot \rho_{23})$$

You may find it useful to derive Equation (5.6.a) from Equation (5.2). In any case, notice the similarity in the patterns of variables between equations (5.6) and (5.6.a).

If fifty securities were to be included in the investor's portfolio, 2500 expressions must be solved and then added for Equation (5.2). This portfolio would require solutions to 1275 expressions for solving the more simple expression (5.6.a). Obviously, as the number of securities in the portfolio becomes large, computers become quite useful in working through the repetitive calculations. The equations are not difficult to solve, they are merely repetitive and time-consuming.

In our first example, the weighted average of the standard deviation of returns of the two securities one and two is 26%, yet the standard deviation of returns of the portfolio they combine to make is only 23%. Clearly, some risk has been diversified away by combining the two securities into the portfolio. In fact, the risk of a portfolio will almost always be lower than the weighted average of the standard deviations of the securities that comprise that portfolio.

For a more extreme example of the benefits of diversification, consider two securities, three and four, whose potential return outcomes are perfectly inversely related. Data relevant to these securities is listed in Table (5.1). If outcome one occurs, security three will realize a return of 30%, and security four will realize a 10% return level. If outcome two is realized, both securities will attain returns of 20%. If outcome three is realized, securities three and four will attain return levels of 10% and 30%, respectively. If each outcome is equally likely to occur ([P_i] is .333 for all outcomes), the expected return level of each security is 20%; the standard deviation of returns for each security is .08165. The expected return of a portfolio combining the two securities is 20% if each security has equal portfolio weight ([w_3] = [w_4] = .5), yet the standard deviation of portfolio returns is zero. Thus, two relatively risky securities have been combined into a portfolio that is virtually risk-free.

TABLE 5.1: Portfolio return with perfectly inversely correlated securities. $W_3 = W_4 = 0.5$

i	R_{3i}	R_{4i}	R_{pi}	P_i
1	.30	.10	.20	.333
2	.20	.20	.20	.333
3	.10	.30	.20	.333

Notice in the previous paragraph that we first combined securities three and four into a portfolio and then found that portfolio's return given each outcome. The portfolio's return is 20% regardless of the outcome; thus, it is risk free. The same result could have been obtained by finding the variances of securities three and four from Equation (4.11), the correlation coefficient between their returns from Equations (4.13) and (4.14), then solving for portfolio variance with Equation (5.5) as in Table (5.2).

TABLE 5.2: Portfolio return with perfectly inversely correlated securities.

Given:

$\overline{R}_3 = 0.20$ $\qquad$ $\overline{R}_4 = 0.20$

$\sigma_3 = 0.08165$ $\qquad$ $\sigma_4 = 0.08165$

$w_3 = 0.50$ $\qquad$ $w_4 = 0.50$

$\rho_{3,4} = -1$

Then:

$$\overline{R}_p = w_3\overline{R}_3 + w_4\overline{R}_4 = (0.5 \times 0.20) + (0.5 \times 0.20) = 0.20$$

$$\sigma_p = \sqrt{w_3^2\sigma_3^2 + w_4^2\sigma_4^2 + 2(w_3 w_4 \sigma_3 \sigma_4 \rho_{34})}$$

$$\sigma_p = \sqrt{0.5^2 \times 0.0066667 + 0.5^2 \times 0.0066667 + 2 \times 0.5 \times 0.5 \times 0.08165 \times 0.08165 \times (-1)}$$

$$\sigma_p = \sqrt{0.0016667 + 0.0016667 - 0.003333} = \sqrt{0} = 0$$

The implication of the two examples provided in this chapter is that security risk can be diversified away by combining the individual securities into portfolios. Thus, the old stock market adage "Don't put all your eggs in one basket" really can be validated mathematically. Spreading investments across a variety of securities does result in portfolio risk that is lower than the weighted average risks of the individual securities. This diversification is most effective when the returns of the individual securities are at least somewhat unrelated; or better still, inversely related as were securities three and four in the previous example. For example, returns on a retail food company stock and on a furniture company stock are not likely to be perfectly positively correlated; therefore, including both of them in a portfolio may result in a reduction of portfolio risk. From a mathematical perspective, the reduction of portfolio risk is dependent on the correlation coefficient of returns (ρ_{ij}) between securities included in the portfolio. Thus, the lower the correlation coefficients between these securities, the lower will be the resultant portfolio risk. In fact, as long as (ρ_{ij}) is less than one, which, realistically is always the case, some reduction in risk can be realized from diversification.

Consider Figure (5.1). The correlation coefficient between returns of securities C and D is one. The standard deviation of returns of any portfolio combining these two securities is a weighted average of the returns of the two securities' standard deviations. Diversification here yields no benefits. In Figure (5.2), the correlation coefficient between returns on Securities A and B is .5. Portfolios combining these two securities will have standard deviations less than the weighted average of the standard deviations of the two securities. Given this lower correlation coefficient, which is more representative of "real world" correlations, there are clear benefits to diversification. In fact, we can see in Figures (5.3) and (5.4) that decreases in correlation coefficients result in increased diversification benefits. Lower correlation coefficients result in lower risk levels at all levels of expected return. Thus, an investor may benefit by constructing his portfolio of securities with low correlation coefficients.

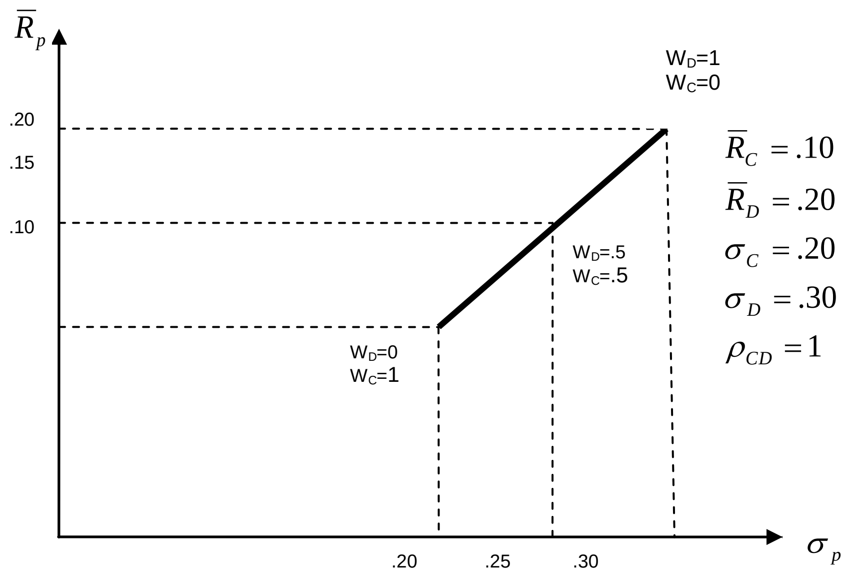

Figure 5.1: Relationship between portfolio return and risk when ρ_{CD}=1

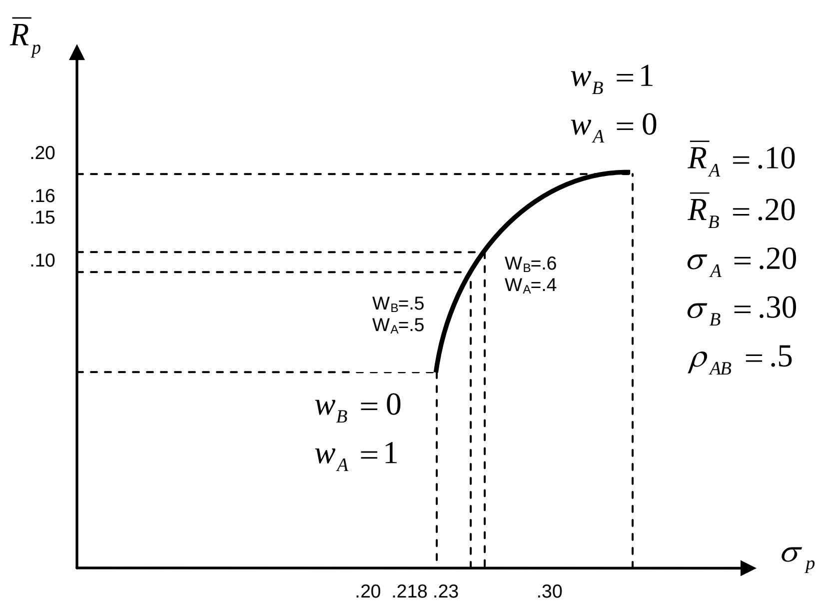

Figure 5.2: Relationship between portfolio return and risk when ρ_{AB}=.5

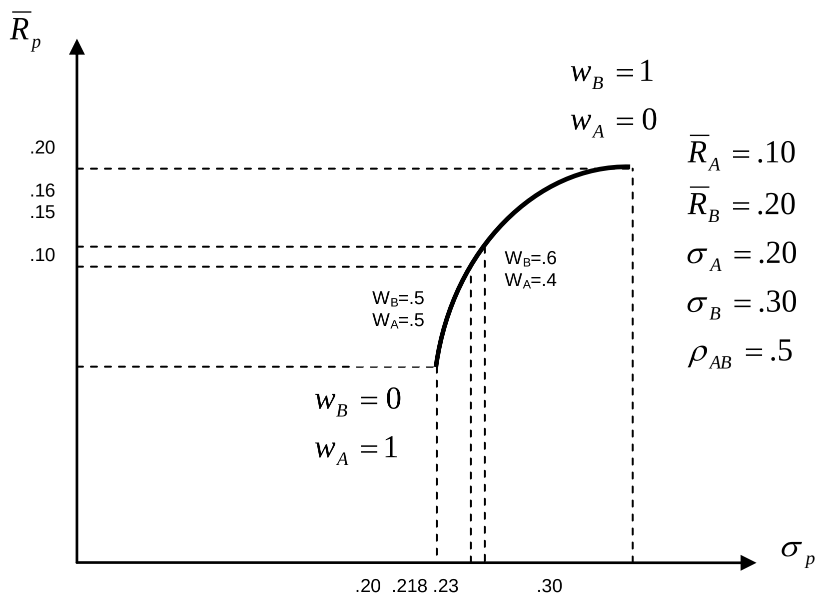

Figure 5.3: The relationship between portfolio return and risk when ρ_{EF}=0

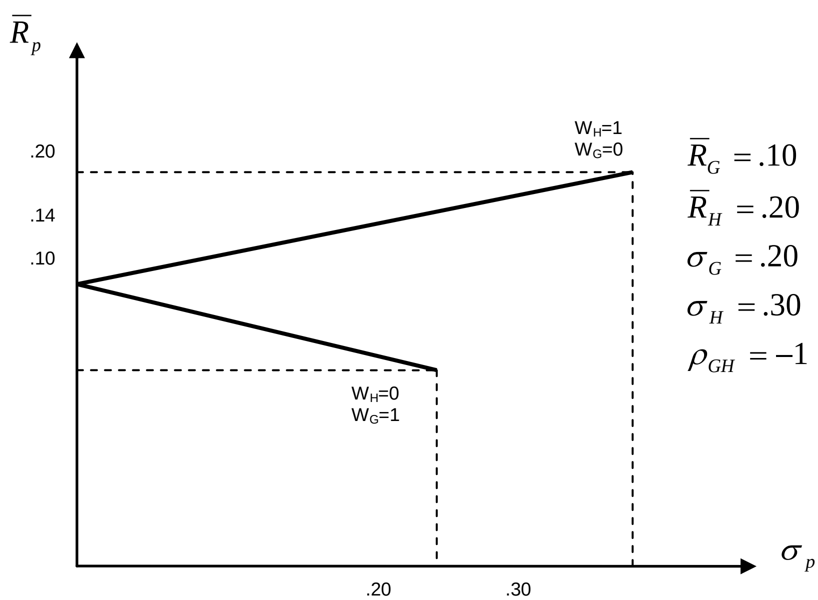

Figure 5.4: The relationship between portfolio return and risk when ρ_{GH}=-1

Derivation Box 5.1

Deriving Portfolio Variance

In Chapter Four, we discussed finding variance of returns based on either potential or actual historical returns. Portfolio variance may also be found as a function of potential or historical portfolio returns. However, it is often useful to express portfolio variance as a function of individual security characteristics. For example we may have estimates of security variance and covariance levels (based on historical estimates) but have no information regarding probabilities to associate with outcomes. Furthermore, it is useful to know exactly how changing portfolio weights will affect portfolio variances.

To derive the variance of portfolio (p) as a function of security variances, covariances and weights, we begin with our standard variance expression as a function of (m) potential portfolio return outcomes (j) and associated probabilities.

(5.11) $$\sigma_p^2 = \sum_{j=1}^{m}(R_{pj} - E[R_p])^2 P_j$$

For the sake of simplicity, let the number of securities (n) in our portfolio equal two. From our portfolio return expression, we may compute portfolio variance as follows:

(A) $$\sigma_p^2 = \sum_{j=1}^{m}(w_1 R_{1j} + w_2 R_{21j} - w_1 E[R_1] - w_2 E[R_2])^2 P_j$$

Next, we complete the square for Equation (A) and combine terms multiplied by the two weights to obtain:

(B) $$\sigma_p^2 = \sum_{j=1}^{m}\left[w_1{}^2 (R_{1j} - E[R_1])^2 + w_2{}^2 (R_{2j} - E[R_2])^2 + 2w_1 w_2 (R_{1j} - E[R_1])(R_{2j} - E[R_2])P_j\right]$$

Next, we bring the summation term inside the brackets:

(C)
$$\sigma_p^2 = \left[w_1{}^2 \sum_{j=1}^{m}(R_{1j} - E[R_1])^2 P_j + w_2{}^2 \sum_{j=1}^{m}(R_{2j} - E[R_2])^2 P_j + 2w_1 w_2 \sum_{j=1}^{m}(R_{1j} - E[R_1])(R_{2j} - E[R_2])P_j\right]$$

We complete our derivation by noting our definitions from Chapter Four for variances and covariances as follows:

(5.6) $$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

The process for deriving variances for larger portfolios would be similar.

5.D. GLOBAL PORTFOLIO DIVERSIFICATION

In their text on portfolio analysis, Elton and Gruber report that the average correlation coefficient between returns on two randomly selected stocks of U.S. corporations is approximately .40.[1] This correlation is substantially higher than the dollar return correlation between randomly selected U.S. stocks and randomly selected stocks from other countries, which is likely to range from about .1 to .35. Since the U.S. stock market comprises somewhere between thirty and forty percent of world stock markets, ample opportunity exists for American investors to diversify their portfolio risk without sacrificing portfolio return.

The domestic investor faces several additional risks investing outside of domestic markets:

- Country Risk: Countries face varying levels of political and economic stability. Foreign return variances will vary in foreign countries. However, country risk between many countries will often be quite low.
- Exchange Risk: Currency exchange is simply the trading or swapping of currencies. The currency exchange rate is simply the number of units of one currency that must be exchanged for another; the exchange rate represents the costs of currencies. The exchange rate between dollars and a foreign currency will certainly affect the dollar denominated return on an investment made in that country. For example, if an American firm invests in the United Kingdom, all of the British profits will be generated in pounds. These pounds must be exchanged for dollars before they can be spent in the U.S. Since the dollar exchange rate (the value of the dollar) varies over time, one cannot be certain exactly how many dollars can be purchased with profits denominated in pounds. This is clearly a source of risk to American investors. On the other hand, the low covariance between values of currencies from different countries will serve to improve overall portfolio diversification.

Although foreign investments are likely to have higher risk (variance) levels for the American investor than the typical domestic investment, they still represent an opportunity for Americans to reduce portfolio risk without sacrificing return. This is due to the particularly low correlation coefficients between American and foreign securities. In fact the following is offered in support of globalizing investment portfolios:

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[1]Elton, Edwin J. and Martin J. Gruber. Modern Portfolio Theory and Investment Analysis, fourth edition. New York: John Wiley & Sons, Inc.: p.252.

1. Portfolio risk at any return level will be lower for a globally diversified portfolio than for a domestic portfolio.
2. Portfolio return at any risk level will be higher for a globally diversified portfolio than for a domestic portfolio.
3. Fewer securities from global markets will be required to attain a given portfolio diversification and risk level than would be required from only a domestic market. This is significant because larger portfolios typically require larger brokerage fees to acquire and are more costly and time consuming to manage.

Elton and Gruber [1992] report that the average correlation coefficient between returns on U.S. securities is approximately .40. The correlation coefficient between two randomly selected 100 security portfolios, one drawn from NYSE stocks and the other selected from AMEX stocks exceeds .90. However, correlations between stock indices of different international markets is significantly smaller than these values, as indicated by the Table 5.3:

TABLE 5.3: **Correlation Coefficients Between Market Indices**

	Aus'l	Au'a	Bel.	Can.	Fra.	Ita.	Jap.	Net.	Swi.	U.K.	W.Ger
Australia											
Austria	.013										
Belgium	.117	.044									
Canada	.167	.058	.179								
France	.082	.069	.177	.163							
Italy	.022	.011	.079	.060	.012						
Japan	.086	.071	.086	.192	.106	.102					
Netherlands	.134	.038	.232	.361	.158	.098	.167				
Switzerland	.173	.045	.164	.289	.148	.174	.192	.283			
U.K.	.171	.034	.093	.146	.039	.078	.110	.131	.002		
W.Germany	.106	.072	.186	.201	.153	.050	.113	.357	.207	.030	
U.S.	.137	.027	.205	.634	.107	.002	.092	.344	.242	.096	.163

Source: Joy, Panton, Reilly and Martin: Financial Review, 1976)

These correlation coefficients are all based on amounts converted into U.S. dollars.

5.E. EFFICIENCY AND DOMINANCE

Because investors prefer as much return and as little risk as possible, the most efficient portfolios are those with the following characteristics:

1. Smaller risk than all portfolios with identical or larger returns and
2. Greater return than all portfolios with identical or less risk.

One portfolio dominates a second when one of the following three conditions is met:

1. the first portfolio has both higher return and smaller risk levels than does the second,

2. both portfolios have identical variance but the first portfolio has a higher return level than does the second, or
3. both portfolios have identical returns but the first portfolio has a smaller variance than does the second.

A portfolio is considered dominant if it is not dominated by any other portfolio. Thus, the most efficient portfolios are all dominant.

5.F. CONSTRUCTION OF THE EFFICIENT FRONTIER

Consider a market where the average coefficient of correlation between returns on securities is .8. (This is not really an unrealistic assumption.) For sake of simplicity, assume that there exist in this market five securities, (A) through (E). Combine securities (A) and (B) into a portfolio. Return and risk combinations of the resultant portfolio will fall somewhere on the curve extending between the two securities, depending on their relative weights (See Figure 5.5). Similarly, securities (B) and (C) can be combined into portfolios as can securities (C) and (D), and (D) and (E) (See Figure 5.6). We have constructed a series of curves representing risk-return combinations of an infinite number of two security portfolios. These resultant portfolios themselves can be combined into additional portfolios. For example, consider portfolios (AB) and (BC) in Figure 5.6. These portfolios can be combined into further portfolios as can portfolios (BC) and (CD) as well as (CD) and (DE). The resultant portfolios can all be combined into additional portfolios. Notice that as the portfolios become more diversified, they become more efficient. Thus, the curves representing the risk-return combinations of these portfolios fall further to the northeast on the risk-return space. However, the benefits of this diversification must reach a limit. This is because the portfolios that are being combined are more correlated than the individual securities that they contain. On the curve indicating this limit, further diversification cannot result in more efficient portfolios. The upward sloping portion of this curve is called the Efficient Frontier. The most efficient portfolios of risky assets will have risk-return combinations falling on the efficient frontier. The Efficient Frontier represents the risk-return combinations of the most efficient portfolios in the feasible region. (The feasible region is simply the risk-return combinations of all portfolios available to investors.) Thus, the Efficient Frontier is the left-most, uppermost boundary of the feasible region.

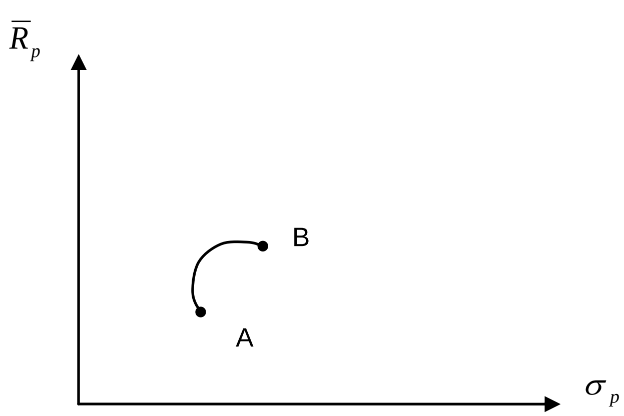

Figure 5.5: Portfolio risk-return levels when ρ_{AB}=.8

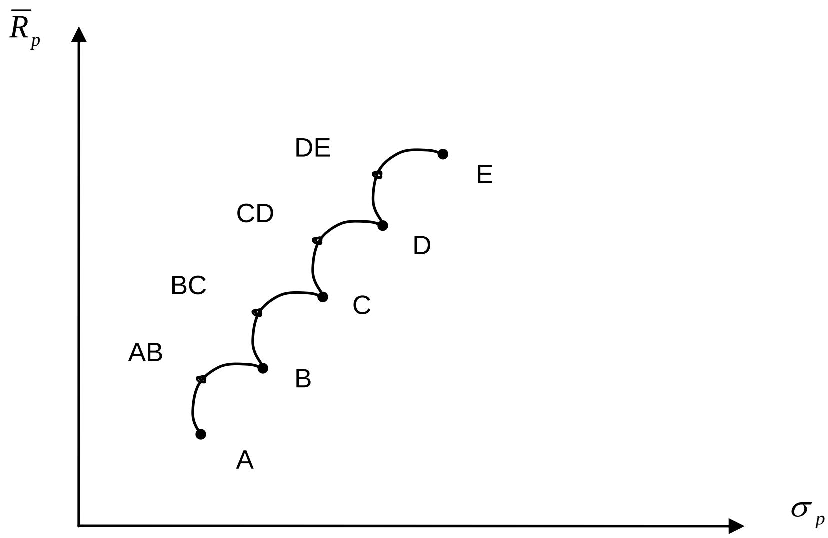

Figure 5.6: Portfolio risk-return levels when ρ_{ij}=.8 for all i and j; portfolios are each comprised of two securities A through E.

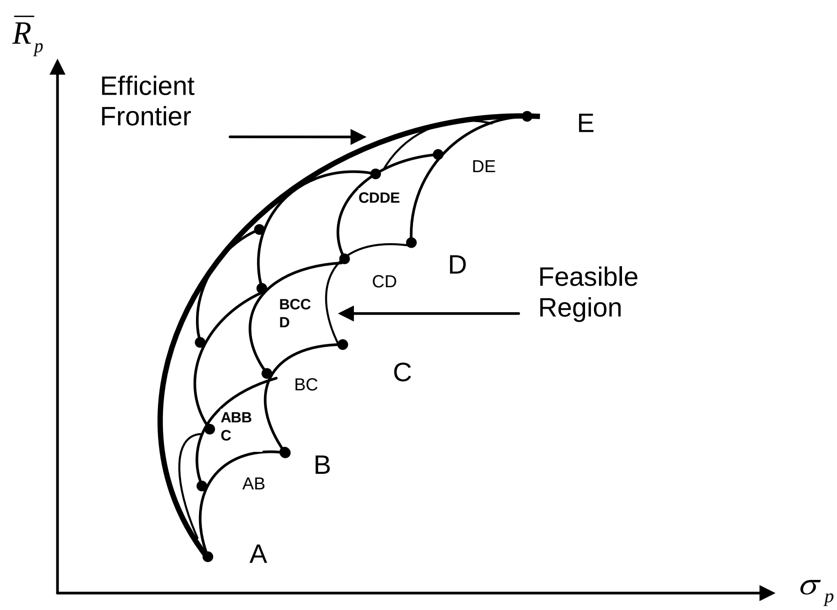

Figure 5.7: Efficient frontier and feasible region

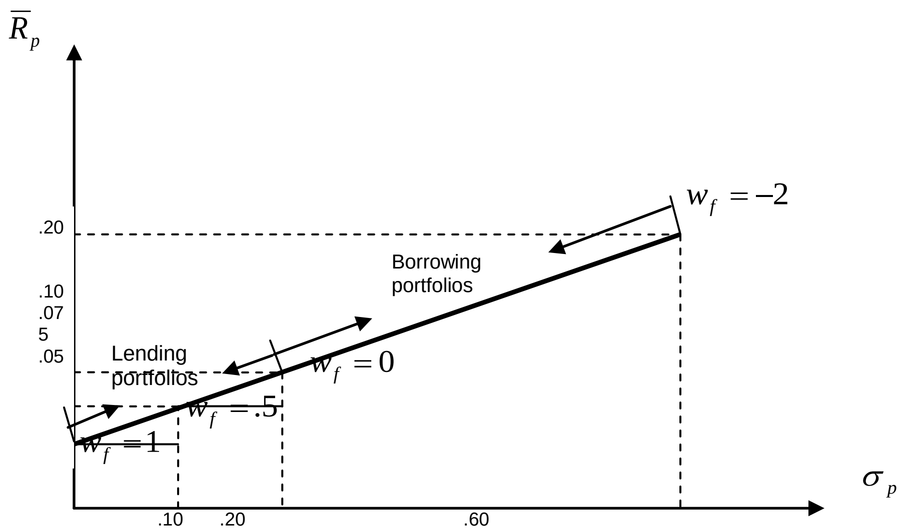

Figure 5.8: Combinations of risky asset portfolio and the risk-free asset

5.G. THE RISK-FREE ASSET

In reality, there exists no risk-free asset. However, for computational purposes, it is useful to assume the existence of such an asset. Historical evidence suggests that short-term United States Treasury bills have been among the most reliable in actually realizing the returns expected by investors. By purchasing treasury bills, an investor is loaning the government money. The United States government has proven to be an extremely reliable debtor (at least it makes good on all of its treasury bills). Treasury bills are fully backed by the full faith and credit of the U.S. government, which has substantial resources due to its ability to tax citizens and create money. Thus, these securities are safer than the safest of corporate bonds or short-term notes. They may even be safer than U.S. F.D.I.C. backed savings accounts and certificates of deposit, which are backed only by the limited resources available to banks and to the F.D.I.C. The resources of the F.D.I.C. are limited to assets it receives from participating commercial and savings banks. (However, Congress has passed a resolution promising to back the F.D.I.C. if it runs out of funds, although this process can be slow and might be uncertain.) Thus, the Treasury bill is practically risk-free and probably the safest of all investments. Because the United States Treasury bill seems to be the safest of all investments, its characteristics are often used as surrogates for the characteristics of the risk-free asset.

By definition, the variance (or, standard deviation) of expected returns on the risk-free asset is zero. Thus, an investor purchasing such an asset will certainly receive the return he originally expected. Though this asset is riskless, the investor will require a return, compensating him for inflation and his time value of money. This risk-free rate of return (r_f) can be approximated with the short-term Treasury bill rate.

The risk-free asset can be combined with any portfolio of risky assets. Such a portfolio will have a risk-return combination which is simply a weighted average of the risky portfolio's and the risk-free asset's risk-return combinations. For example, consider a portfolio of risky assets with expected return and standard deviation levels of 10% and 20% and a risk-free asset with an expected return of 5%. If the portfolio and the risk-free asset were combined into a new portfolio with equal weights ($w_f = w_m = .5$), the resultant portfolio would have expected return and standard deviation levels of 7.5% and 10%:

$$E\lfloor R_p \rfloor = (.5 \cdot .10) + (.5 \cdot .05) = .75$$

$$\sigma_p = \sqrt{(.5^2 \cdot .20^2) + (.5^2 \cdot 0^2) + 2(.5 \cdot .5 \cdot .1 \cdot 0 \cdot 0)}$$

$$\sigma_p = \sqrt{(.25 \cdot .04) + (0) + (0)} = .10.$$

Notice that the correlation coefficient between returns on any risky asset and the risk-free asset must be zero. Thus, both portfolio expected returns and portfolio standard deviations will be a linear combination of the individual security returns and standard deviations only when (ρ_{ij}) = 1 or, as in this case, when a risk-free asset is combined with a risky investment.

If an investor has the opportunity to borrow money at the risk-free rate of return (r_f), he has the opportunity to create a negative weight (w_f) for the risk-free asset. For example, if an investor had an initial wealth level of $1000, but wished to invest $3000 in a risky asset with an expected return of 10%, he could borrow $2000 at the risk-free rate of 5% if the lender were certain the investor would fulfill his debt obligation. Since the investor is borrowing money rather than lending (buying Treasury bills is, in effect, lending the government money), the weight associated with the risk-free asset is negative. Because the total sum invested in the risky asset is three times as great as the investor's initial wealth level, (w_A) is equal to 3. The investor's expected portfolio return level is 20%, higher than the return of either of the assets comprising the portfolio:

$$E\lfloor R_p \rfloor = (-2 \cdot .05) + (3 \cdot .10) = .20$$

Notice that the sum borrowed is twice as great as the investor's initial wealth level, thus (w_f) is equal to -2. The standard deviation of returns on the portfolio is .6:

$$\sigma_p = \sqrt{(-2^2 \cdot .0^2) + (3^2 \cdot 20^2) + 2(-2 \cdot 3 \cdot 0 \cdot .20 \cdot 0)}$$

$$\sigma_p = \sqrt{(3^2 \cdot .20^2)} = \sqrt{.36} = .6.$$

Notice that the portfolio standard deviation is higher than the standard deviations of either of the assets comprising the portfolio. Therefore, borrowing money (creating leverage) permits the investor to increase his expected returns; however, he must also face additional risk. Whether an investor will borrow, and exactly how much he will borrow will be determined later in this chapter.

Consider an investor who has the opportunity to invest in a combination of a risk-free asset and one of several risky portfolios (A) through (E) depicted in Figure 5.9. Which of these five portfolios is the best to combine with the risk-free asset? Notice that the portfolios with risk-return combinations on the line connecting the risk-free asset and portfolio (C) dominate all other portfolios available to the investor. Thus, any portfolio whose risk-return combination falls on lines extending through portfolios (A), (B), (D), and (E) will be dominated by some portfolio whose risk-return combination is depicted on the line extending through portfolio(C). This line has a steeper slope than all other lines between the risk-free asset and risky portfolios. The investor's objective is to choose that portfolio of risky assets enabling him to maximize the slope of this line; that is, the investor should pick that portfolio with the largest possible ($\square_p$), where ($\square_p$) is defined by Equation (5.7):

$$\frac{E\lfloor R_p \rfloor - r_f}{\sigma_p} = \Theta_p \quad . \tag{5.7}$$

Therefore, the investor should invest in some combination of portfolio (C) and the risk-free asset. If the curve connecting portfolios (A) through (E) were the Efficient Frontier, then portfolio (C) would be referred to as the market portfolio. This is because every risk averse investor in the market should select this portfolio of risky assets to combine with the riskless asset. Notice that the line extending through portfolio (C) is tangent to the curve at point (C).

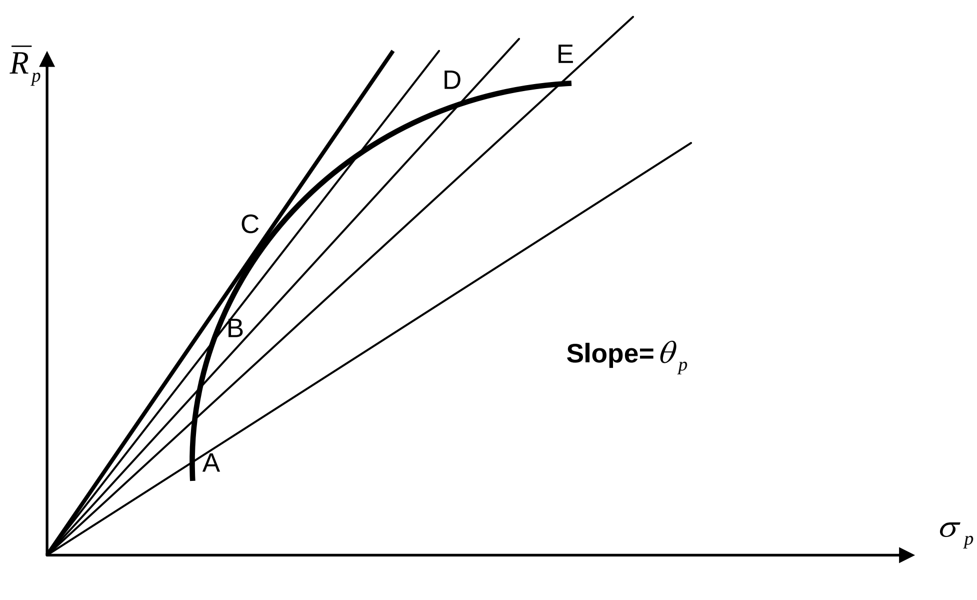

Figure 5.9: Combination of risk-free asset with one of five portfolios of risky assets

5.H. THE CAPITAL MARKET LINE

The best portfolio of risky assets to combine with the risk-free security lies on the Efficient Frontier, tangent to the line extending from the risk-free security. This line is referred to as the Capital Market Line (CML). Notice that portfolios on the Capital Market Line dominate all portfolios on the Efficient Frontier. If a risk-free security exists, the Capital Market Line represents risk-return combinations of the best portfolios of securities available to investors. Thus, an investor's risk-return combinations are constrained by the Capital Market Line.

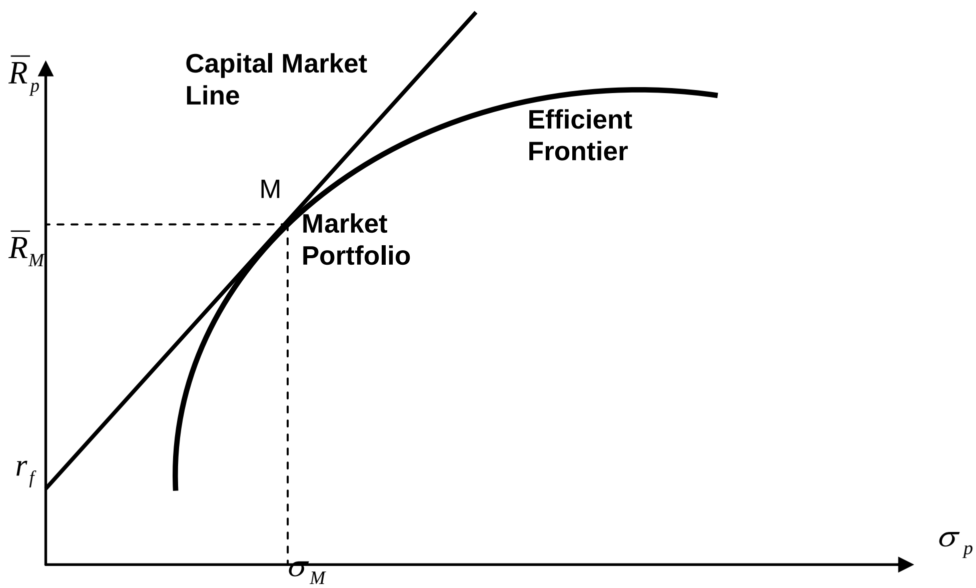

Figure 5.10: The Capital Market Line

The most efficient portfolio on the Efficient Frontier to combine with the riskless asset is referred to as the Market Portfolio (depicted by [M] in Figure 5.10). Thus, the Market Portfolio lies at a point of tangency between the Efficient Frontier and the Capital Market Line. All investors should hold portfolios of risky assets whose weights are identical to those of the Market Portfolio. The Capital Market Line combines the Market Portfolio with the riskless asset. This line can be divided into two parts: the lending portion and the borrowing portion. If an investor invests at point (M) on the Capital Market Line, all of his money is invested in the Market Portfolio. If he invests to the left of (M), his portfolio is a lending portfolio. That is, he has purchased treasury bills, in effect, lending the government money, and invested the remainder of his funds in the Market Portfolio. If he invests to the right of point (M), he has a borrowing portfolio. In this case, he has invested all of his funds in the Market Portfolio and borrowed additional money at the risk-free rate to invest in the Market Portfolio. All investors will invest at some risk-return combination on the Capital Market Line. Exactly which risk-return combination an investor will choose will depend on the investor's level of risk aversion.

5.I. INTRODUCTION TO THE CAPITAL ASSET PRICING MODEL

The Capital Asset Pricing Model (CAPM) provides us with a theory of equilibrium in capital markets. This means that the CAPM will explain how investors price securities in the marketplace, as long as the assumptions underlying the theory are fulfilled. The assumptions that underlie the CAPM are as follows:

1. Capital markets are perfectly efficient. This means that security prices fully reflect all available information at all times. Characteristics of perfectly efficient capital markets include:
    - zero transactions costs; that is, investors, corporations and institutions buy and sell securities without incurring brokerage or other transactions fees.
    - no taxes on investment income.
    - investors have equal access to information on a costless basis.
    - no single investor's transactions can influence the market price of any security.
2. Security returns are normally distributed implying that investors who maximize expected utility focus only on expected return and risk levels of their portfolios.
3. All assets are marketable and infinitely divisible.
4. No restrictions are placed on short-sales (borrowing securities, selling them and repurchasing later).
5. Investors all have identical expectations regarding security expected return and risk levels.
6. There exists a risk free security with no restrictions on borrowing and lending.
7. Investor planning horizons are for a single time period.

If these assumptions hold, we can use the Capital Asset Pricing Model to estimate the risk of an investment. This risk measure can then be used to calculate a discount rate for computing the present value of that investment. The purchase/sale decision is made on the investment's present value relative to the investment's market price.

5.J. SYSTEMATIC AND UNSYSTEMATIC RISK

The risk associated with a security might be classified as either systematic or unsystematic risk. Systematic risk is that portion of a security's risk that is related to variance of the market portfolio. Systematic risk is sensitivity to market portfolio fluctuations. Thus, systematic risk is often referred to as market-related risk. Unsystematic risk is that portion of a security's variance that is unrelated to risk of the market portfolio; that is, unsystematic risk (or firm-specific or unique risk) is unique to the security under analysis. For example, a labor strike or death of the company's chief executive officer may affect firm performance and may be regarded as a firm-specific or unsystematic risk.

Figure 5.11 depicts the impact of portfolio size (number of securities) on the risk of a randomly selected portfolio of stocks. The average standard deviation of a randomly selected stock from the New York Stock Exchange is approximately .40. About half of this risk might be regarded as being market-related and the remainder is firm-specific. As additional randomly selected stocks are added to this equally weighted portfolio, portfolio risk declines, quickly at first but then at a slower rate as the portfolio becomes better diversified. The diversification eliminates approximately 99% of firm-specific risk with investment in as few as 30 securities, reducing overall risk to market-related only. The standard deviation of the market portfolio is approximately 20%.

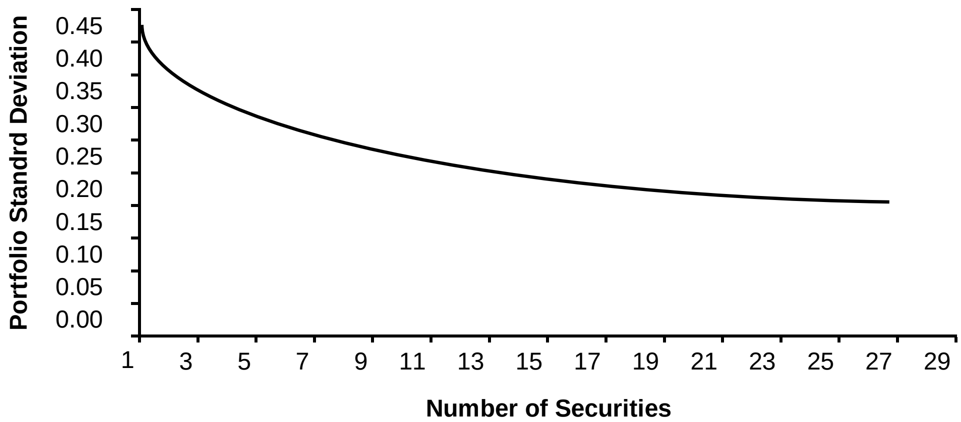

Figure 5.11: Portfolio Size and Risk Reduction

The required return of any security will be related to a risk-free component compensating investors for inflation and their time values of money and premiums compensating investors for both market related and unique risk (though, as we shall see later, unique risk compensation will be zero for a perfectly diversified portfolio). Market related risk of a security can be measured as the standard deviation of returns associated with that security relative to the standard deviation of returns associated with the market portfolio multiplied by the coefficient of correlation between returns on the security and the market portfolio:

(5.8) $$\beta_i = \frac{\sigma_j}{\sigma_m} \cdot \rho_{i,m} \qquad .$$

Thus, Beta (β_i) measures the risk of security (i) relative to the risk of the market portfolio. The coefficient of correlation acts as a sort of "fudge factor" relating to the reduction in portfolio risk realized by including security (i) in a well- diversified portfolio. If the standard deviation of expected returns on security (A) and the market portfolio were .4 and .2, respectively, and the correlation coefficient between returns on the two were .75, the Beta (β_A) of security (A) would be 1.5:

$$\beta_i = \frac{\sigma_A}{\sigma_m} \cdot \rho_{A,m} = \frac{.4}{.2} \cdot .75 = 1.5 \qquad .$$

Notice that Equation (5.8) can be rewritten as follows:

(5.9) $$\beta_i = \frac{\sigma_i \sigma_M \rho_{iM}}{\sigma_m{}^2} = \frac{COV(i,M)}{\sigma_m{}^2}$$

Betas are normally computed on the basis of historical standard deviations and correlation coefficients. Many investors believe that the relative stability over time of these statistical measures justifies the use of "historical" betas. Historical betas are computed on the basis of historical standard deviations and correlation coefficients. Frequently, analysts will compute historical stock betas, standard deviations and correlation coefficients based on five years of historical monthly returns data. Several investment advisory services such as Value Line will provide historical betas for a large number of widely traded stocks.

By definition, the market portfolio (or "average" security) requires a risk premium of (r_m - r_f). Therefore, an investor requires this premium to compensate for the risk associated with the "average" security. In fact, the systematic risk premium required by any investor for any security (i) is:

(5.10) $$(\mathrm{rr}_i - r_f) = \beta_i (r_m - r_f) \quad .$$

Notice that, since $(\sigma_{m,m}) = (\sigma_m^2)$, and $(\rho_{m,m}) = 1$, the beta of the market portfolio equals one.

If an investor assumes additional unsystematic (unique) risk by the purchase of a security, he will require an unsystematic risk premium. This premium will be unrelated to the market portfolio; thus, it will be unique to each individual security. The total risk premium required by an investor for the purchase of security (A) will be:

(5.11) $$(\mathrm{rr}_A - r_f) = \beta_A (r_m - r_f) + \mu_A$$

Covariances between the non-market related return components between securities in a well-diversified portfolio will, on average, equal zero. We find, using time-series data, that on average, securities earn returns equal to the risk-free return plus their required systematic risk premium components. Therefore, the unsystematic risk premium required by an investor holding a well-diversified portfolio will be zero. We can ignore the (μ_A) component of Equation (5.11) for shareholders with well-diversified portfolios. Therefore, the return required by any investor to purchase any security (i) will be:

(5.12) $$\mathrm{rr}_i = r_f + \beta_i (r_m - r_f)$$

This is the Capital Asset Pricing Model. The CAPM enables us to determine the required return for any investment, given its risk characteristics and the current risk-free rate of return. Thus, an investor must expect to receive the required return on an investment in order to purchase it. If capital markets function according to the assumptions outlined in Section 5.I, security prices will be such that the expected returns on all securities will equal their required returns. Although many of these assumptions do not hold in reality, the CAPM still provides us with good approximations of required security returns. Nonetheless, there exist a number of variations of the CAPM that have been adjusted to adapt to more realistic market conditions.

If the required return on the market portfolio were .14 and the current risk-free rate were .06, the required return for security (A) described in Section (5.J) would be .18:

$$rr_A = .06 + 1.5(.14 - .06) = .18 \quad .$$

5.K. RISK-ADJUSTED DISCOUNT RATES

The required rate of return generated by the Capital Asset Pricing Model provides an excellent risk-adjusted discount rate useful for evaluating a variety of investments. This risk-adjusted discount rate reflects inflation and investors' time value of money through its risk-free component. The CAPM reflects investors' risk-return preferences through the market systematic risk premium (r_m - r_f). Furthermore, the model reflects the risk of the security under evaluation through the Beta (β_i) component. Therefore, any security can be evaluated by determining its Beta (β_i), plugging it into the CAPM, and then discounting projected cash flows using the required rate of return as a discount rate:

(5.8) $$\beta_i = \frac{\sigma_j}{\sigma_m} \cdot \rho_{i,m}$$

(5.12) $$\mathrm{rr_i} = r_f + \beta_i (r_m - r_f)$$

(5.13) $$PV_i = \sum_{t=1}^{n} \frac{CF_t}{(1 + rr_i)^t} \quad .$$

Thus, if security (A) were expected to pay a $10 dividend in each of the next three years and then be sold for $100, its current value would be $82.61:

$$PV_A = \frac{\$10}{(1+.18)^1} + \frac{\$10}{(1+.18)^2} + \frac{\$10 + \$100}{(1+.18)^3} = 82.61$$

CHAPTER FIVE QUESTIONS AND PROBLEMS

5.1. An investor is considering combining Douglas Company and Tilden Company common stock into a portfolio. Fifty percent of the dollar value of the portfolio will be invested in Douglas Company stock; the remainder will be invested in Tilden Company stock. Douglas Company stock has an expected return of six percent and an expected standard deviation of returns of nine percent. Tilden Company stock has an expected return of twenty percent and an expected standard deviation of thirty percent. The coefficient of correlation between returns of the two securities has been shown to be .4. Compute the following for the investor's portfolio:

a. expected return
b. expected variance
c. expected standard deviation

5.2. Work through each of your calculations in Problem 5.1 again assuming the following weights rather than those given originally:

a. 100% Douglas Company stock; 0% Tilden Company stock
b. 75% Douglas Company stock; 25% Tilden Company stock
c. 25% Douglas Company stock; 75% Tilden Company stock
d. 0% Douglas Company stock;100% Tilden Company stock

5.3. How do expected return and risk levels change as the portfolio proportions invested in Tilden Company stock increase? Why? Prepare a graph with expected portfolio return on the vertical axis and portfolio standard deviation on the horizontal axis. Plot the expected returns and standard deviations for each of the portfolios whose weights are defined in Problems 5.1 and 5.2. Describe the slope of the curve connecting the points on your graph.

5.4. The common stocks of the Landon Company and the Burr Company are to be combined into a portfolio. The expected return and standard deviation levels associated with the Landon Company stock are five and twelve percent, respectively. The expected return and standard deviation levels for Burr Company stock are ten and twenty percent. The portfolio weights will each be 50%. Find the expected return and standard deviation levels of this portfolio if the coefficient of correlation between returns of the two stocks is:

a. 1
b. .5
c. 0
d. -.5
e. -1

5.5. Describe how the coefficient of correlation between returns of securities in a portfolio affect the return and risk levels of that portfolio.

5.6. An investor is considering combining Securities A and B into an equally weighted portfolio. This investor has determined that there is a twenty percent chance that the economy will perform very well, resulting in a thirty percent return for security A and a twenty percent for security B. The investor estimates that there is a fifty percent chance that the economy will perform only adequately, resulting in twelve percent and ten percent returns for Securities A and B. The investor estimates a thirty percent probability that the economy will perform poorly, resulting in a negative nine percent return for Security A and a zero percent return for Security B. These estimates are summarized as follows:

outcome	probability	R_{ai}	R_{bi}
1	.20	.30	.20
2	.50	.12	.10
3	.30	-.09	0

a. What is the portfolio return for each of the potential outcomes?
b. Based on each of the outcome probabilities and potential portfolio returns, what is the expected portfolio return?
c. Based on each of the outcome probabilities and potential portfolio returns, what is the standard deviation associated with portfolio returns?
d. What are the expected returns of each of the two securities?
e. What are the standard deviation levels associated with returns on each of the two securities?
f. What is the coefficient of correlation between returns of the two securities?
g. Based on your answers to part d in this problem, find the expected portfolio return. How does this answer compare to your answer in part b?
h. Based on your answers to parts e and f, what is the expected deviation of portfolio returns? How does this answer compare to your answer in part c?

5.7. An investor has combined securities X, Y and Z into a portfolio. He has invested $1000 in Security X, $2000 into Security Y and $3000 into Security Z. Security X has an expected return of 10%; Security Y has an expected return of 15% and security Z has an expected return of 20%. The standard deviations associated with Securities X, Y and Z are 12%, 18% and 24%, respectively. The coefficient of correlation between returns on Securities X and Y is .8; the correlation coefficient between X and Z returns is .7; the correlation coefficient between Y and Z returns is .6. Find the expected return and standard deviation of the resultant portfolio.

5.8. An investor wishes to combine Stevenson Company stock and Smith Company stock into a riskless portfolio. The standard deviations associated with returns on these stocks are 10% and 18% respectively. The coefficient of correlation between returns on these two stocks is -1. What must be each of the portfolio weights for the portfolio to be riskless?

5.9. Assume that the coefficient of correlation between returns on all securities equals zero in a given market. There are an infinite number of securities in this market, all of which have the same standard deviation of returns (assume that it is .5). What would be the portfolio return standard deviation if it included all of these infinite number of securities in equal investment amounts? Why? (Demonstrate your solution mathematically.)

5.10. Which of the following portfolios are dominant?

Portfolio	Expected Return	Standard Deviation
a	.05	0

b	.07	.05
c	.11	.15
d	.15	.04
e	.18	.10
f	.19	.35

5.11. Portfolios X and Y are dominant portfolios from which the Efficient Frontier can be constructed. Portfolio X has an expected return of 6% and a standard deviation of 5%. Portfolio Y has an expected return of 12% and a standard deviation of 15%. The coefficient of correlation between returns on these portfolios is (-.5). Construct a graph with expected return and standard deviation axes and plot the coordinates for portfolios with the following weights:

$w_x = 1$ $\quad w_y = 0$
$w_x = .75$ $\quad w_y = .25$
$w_x = .5$ $\quad w_y = .5$
$w_x = .25$ $\quad w_y = .75$
$w_x = 0$ $\quad w_y = 1$

a. Construct the Efficient Frontier for this security market. (Note: Remember that only these two portfolios are required for the Efficient Frontier construction since they are the only dominant portfolios.)

b. Based on your graph of the Efficient Frontier and a five percent risk-free rate of return, estimate the expected return and standard deviation levels of the market portfolio.

c. Based on your solutions to parts a and b, construct the Capital Market Line.

d. What is the slope of the Capital Market Line in Part c?

5.12. Are extremely risk averse investors likely to be borrowers or lenders? How does risk aversion affect borrowing levels? Why?

5.13. Given that correlation coefficients between domestic securities exceed correlation coefficients between domestic and foreign securities, how would expanding the feasible region to include foreign securities affect the Efficient Frontier? How would this expansion affect the Capital Market Line?

5.14. A stock currently selling for $60 has a historical standard deviation of .25 and a coefficient of correlation with the market portfolio of .4. Over the same period, the historical standard deviation of the market portfolio was .16. Investors anticipate dividends of $2 per share in one year at which time the stock can be sold for $65. Determine the following for the stock if the current Treasury Bill rate is .05 and the required return on the market portfolio is .12:

a. The stock Beta.
b. The required return of the stock.
c. The discount rate to be associated with cash flows from the stock.
d. The present value of cash flows associated with the stock.
e. Whether the stock constitutes a good investment.

5.15. Historical returns for Holmes Company stock, Warren Company stock and the market portfolio along with Treasury Bill (T-Bill) rates are summarized in the following chart:

Year	Holmes Co.	Warren Co.	Market	T-Bill
1996	12%	4%	10%	6%
1997	18%	20%	14%	6%
1998	7%	2%	6%	6%
1999	3%	-3%	2%	6%
2000	10%	9%	8%	6%

a. Calculate return standard deviations for each of the stocks and the market portfolio.
b. Calculate correlation coefficients between returns on each of the stocks and returns on the market portfolio.
c. Prepare graphs for each of the stocks with axes (R_{it} - R_{ft}) and (R_{mt} - R_{ft}) where R_{it} is the historical return in year (t) for stock (i) ; R_{mt} and R_{ft} are historical market and risk-free returns in time (t). Plot regression lines for each of the two stocks; that is, try to fit a single line for each stock that is as close as possible to all of the data points.
d. Calculate Betas for each of the stocks. How do your Betas compare to the slopes of the regression lines that you drew?

5.16. Consider the Holmes and Warren stocks whose historical returns are given in Problem 5.15. Assume an investor had combined each of the stocks into a portfolio such that half of his wealth was invested in each of the stocks at the beginning of each year. Calculate the following for the investor's portfolio:

a. Historical returns for each of the five years.
b. Historical portfolio standard deviation for the five year period.
c. Historical correlation coefficient between the market portfolio and the investor's portfolio.
d. The portfolio Beta.
e. How does this portfolio Beta compare to the Betas of the individual stocks?

5.17. How might a corporation calculate a Beta for one of its assets? Why might this calculation be somewhat more difficult than calculating a stock Beta?

5.18. Calculate the Beta of a risk-free asset.

5.19. If investors can diversify away unsystematic risk by constructing portfolios, why are corporate managers so concerned with the riskiness of their individual firm's operations?