Chapter 4 Return and Risk

The objectives of this chapter are to enable you to:

! *Understand and calculate returns as a measure of economic efficiency*
! *Understand the relationships between present value and IRR and YTM*
! *Understand how obtain an expected security return from potential returns and associated probabilities*
! *Define and measure risk*
! *Understand and measure co-movement*

4.A. INTRODUCTION

The purpose of measuring investment returns is simply to determine the economic efficiency of an investment. Thus, an investment's return will express the profits generated by an initial cash outlay relative to the amount of that outlay. There exist a number of methods for determining the return of an investment. The measures presented in this chapter are return on investment and internal rate of return. Arithmetic and geometric mean rates of return on investment will be discussed along with internal rate of return and bond return measures. These methods differ in their ease of computation and how they account for the timeliness and compounding of cash flows.

4.B. RETURN ON INVESTMENT: ARITHMETIC MEAN

Perhaps the easiest method to determine the economic efficiency of an investment is to add all of its profits (π_t) accruing at each time period (t) and dividing this sum by the amount of the initial cash outlay (P_0). This measure is called a holding period return. To ease comparisons between investments with different life expectancies, one can compute an arithmetic mean return on investment (ROI) by dividing the holding period return by the life expectancy of the investment (n) as follows:

(4.1)
$$ROI_A = \frac{\sum_{t=1}^{n} \pi_t}{P_0} \div n$$

The subscript (A) after (ROI) designates that the return value expressed is an arithmetic mean return and the variable (π_t) is the profit generated by the investment in year (t). Since it is not always clear exactly what the profit on an investment is in a given year, one can compute a return based on periodic cash flows. Therefore, this arithmetic mean rate of return formula can be written:

$$(4.2) \qquad ROI_A = \frac{\sum_{t=0}^{n} CF_t}{n \cdot P_0} = \frac{\sum_{t=1}^{n} CF_t}{n \cdot P_0} - \frac{1}{n}$$

Notice that the summation in the first expression begins at time zero, ensuring that the initial cash outlay is deducted from the numerator. (The cash flow [CF_0] associated with any initial cash outlay or investment will be negative.) The primary advantage of Equation (4.2) over (4.1) is that a profit level need not be determined each year for the investment; that is, the annual cash flows generated by an investment do not have to be classified as to whether they are profits or merely return of capital. Multiplying (P_0) by (n) in the denominator of (4.2) to annualize the return has the same effect as dividing the entire fraction by (n) as in (4.1). In the second expression, the summation begins at time one. The initial outlay is recognized by subtracting one at the end of the computation. For example, consider a stock whose purchase price three years ago was $100. This stock paid a dividend of $10 in each of the three years and was sold for $130. If time zero is the stock's date of purchase, its arithmetic mean annual return is 20%:

$$ROI_A = \frac{-100+10+10+10+130}{3 \cdot 100} = \frac{60}{300} = .20$$

Identically, the stock's annual return is determined by (4.3):

$$(4.3) \qquad ROI_A = \frac{\sum_{t=1}^{n} DIV_t}{n \cdot P_0} + \frac{P_n - P_0}{n \cdot P_0}$$

where (DIV_t) is the dividend payment for the stock in time (t), (P_0) is the purchase price of the stock and (P_n) is the selling price of the stock. The difference (P_n - P_0) is the capital gain realized from the sale of the stock.

Consider a second stock held over the same period whose purchase price was also $100. If this stock paid no dividends and was sold for $160, its annual return would also be 20%:

$$ROI_A = 0 + \frac{160-100}{3 \cdot 100} = \frac{60}{300} = .20$$

Therefore, both the first and second stocks have realized arithmetic mean returns of 20%. The total cash flows generated by each, net of their original $100 investments, is $60. Yet, the first stock must be preferred to the second since its cash flows are realized sooner. The arithmetic mean return (ROI_a) does not account for the timing of these cash flows. Therefore, it evaluates the two stocks

Application 4.1: Fund Performance

This application is concerned with how one might collect price data for a fund and compute returns from that data. Suppose that one is interested in a given publically traded fund for which prices and dividends are reported in standard news sources. First, retrieve fund price and dividend data for each period under consideration. This is easily done with the *Wall Street Journal* or *Yahoo Finance* as well as a variety of online data sources. Calculate periodic holding returns based on the following:

$$r_t = \frac{P_t - P_{t-1} + DIV_t}{P_{t-1}}$$

Date	t	P_t	P_{t-1}	DIV_t	r_t	NOTES
June 30	0	50	-	0	-	First Month
July 31	1	55	50	0	.100	(55/50) - 1 = .10
Aug. 31	2	50	55	0	-.091	(50/55) - 1 = -.091
Sep. 30	3	54	50	0	.080	(54/50) - 1 = .08
Oct. 31	4	47	54	2	-.092	ex-$2 dividend; [(47+2)/54)]-1= -.092
Nov. 30	5	51	47	0	.081	(51/47) - 1 = .081

Now, 5 monthly holding period returns have been computed for the fund. The 5-month holding period return might be approximated as the sum of individual monthly returns (though we will discuss some problems with this in Section 4.B and in Application 4.2). The holding period and arithmetic mean returns for the fund can be computed with an alternative to Equations 4.1 to 4.3 as follows:

$$ROI_H = \frac{\sum_{t=1}^{n} r_t}{n}$$

$$ROI_A = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{.10 - .091 + .08 - .092 + .081}{5} = .0156$$

identically even though the first should be preferred to the second. Because this measure of economic efficiency does not account for the timeliness of cash flows, another measure must be developed.

4.C. RETURN MEASUREMENT: GEOMETRIC MEAN

The arithmetic mean return on investment does not account for any difference between dividends (intermediate cash flows) and capital gains (profits realized at the end of the investment holding period). That is, ROI_A does not account for the time value of money or the ability to re-invest cash flows received prior to the end of the investment's life. In reality, if an investor receives profits in the form of dividends, he has the option to re-invest them as they are received. If profits are received in the form of capital gains, the investor must wait until the end of his investment holding period to re-invest them. The difference between these two forms of profits can be accounted for by expressing compounded returns. That is, the geometric mean return on investment will account for the fact that any earnings that are retained by the firm will be automatically re-invested, thus compounded. If returns are realized only in the form of capital gains, the geometric mean rate of return is computed as follows:

(4.4) $$ROI_g = \sqrt[n]{P_n / P_o} - 1$$

For example, the geometric mean return on the second stock whose price increased from $100 to $160 discussed in Section 3.B is 16.96%, determined as follows:

$$ROI_g = \sqrt[3]{160/100} - 1 = \sqrt[3]{1.6} - 1 = .1696$$

If dividends or intermediate cash flows from the security are realized before the end of the holding period, returns r_t should be computed for each period t and then averaged as follows:

$$r_t = \frac{P_t - P_{t-1} + DIV_t}{P_{t-1}}$$

(4.5) $$ROI_g = \sqrt[n]{\prod_{t=1}^{n}(1+r_t)} - 1$$

Suppose that the stock in our previous example paid $20 in dividends in each of the three years of the holding period rather than generating a $60 capital gain over the three year period. The return r_t for each period would be 20% and the geometric mean return for the stock would be 20% computed as follows:

$$ROI_g = \sqrt[n]{\prod_{t=1}^{n}(1+r_t)} - 1 = \sqrt[3]{\prod_{t=1}^{3}(1+r_t)} - 1 \quad = \sqrt[3]{(1+.2)(1+.2)(1+.2)} - 1 = .20$$

Note that the geometric mean return is higher if profits can be withdrawn from the investment during the holding period.

Application 4.1: Fund Performance (Continued)

This application continues our discussion of how one might collect price data for funds, compute returns from that data and compare funds. Here, we will compare the prices and returns of this fund (we will call it Fund A) to the monthly returns and prices to a second Fund B:

		Fund A				Fund B			
Date	t	P_t	P_{t-1}	DIV_t	r_t	P_t	P_{t-1}	DIV_t	r_t
June 30	0	50	-	0	-	50	-	0	-
July 31	1	55	50	0	.100	80	50	0	.60
Aug. 31	2	50	55	0	-.091	40	80	0	-.50
Sep. 30	3	54	50	0	.080	60	40	0	.50
Oct. 31	4	47	54	2	-.092	30	60	0	-.50
Nov. 30	5	51	47	0	.081	45	30	0	.50

Recall that the arithmetic mean return on investment for Fund A is .0156. The arithmetic mean return on investment on Fund B is .12, computed as follows:

$$ROI_B = \frac{\sum_{t=1}^{n} r_t}{n} = \frac{.60 - .50 + .50 - .50 + .50}{5} = .12$$

This computational procedure is obviously quite misleading. Fund B paid no dividends and ended the five-year period worth $5 less than at the beginning of the five-year period. Despite the fact that Fund B obviously lost money, its arithmetic mean return was computed to be .12 over the five years. Fund A clearly outperformed Fund B, yet its arithmetic mean return appears to be lower. Alternatively, one could compute the arithmetic mean rate of return for Fund B using the following:

$$ROI_B = \frac{P_n - P_0}{n \cdot P_0} = \frac{45 - 50}{5 \times 50} = -.02$$

While this seems more intuitively acceptable, the return for Fund A cannot be computed exactly the same way. However, the geometric mean return on investment can be computed for both funds:

$$ROI_g = \sqrt[n]{\prod_{t=1}^{n}(1 + r_t)} - 1$$

$$ROI_g \ \ for \ \ A = \sqrt[5]{\prod_{t=1}^{5}(1 + r_t)} - 1 \quad = \sqrt[5]{(1 + .1)(1 - .091)(1 + .08)(1 - .092)(1 + .081)} - 1 = .012$$

$$ROI_g \ \ for \ \ B = \sqrt[5]{\prod_{t=1}^{5}(1+r_t)} - 1 \quad = \sqrt[5]{(1+.6)(1-.5)(1+.5)(1-.5)(1+.5)} - 1 = -.0208$$

With the geometric mean return computation, changes in the funds' investment bases are accounted for. That is, as the values of the funds change, the amounts on which returns are computed vary. Hence, the geometric mean rate of return can account for compounding and the comparison of fund returns is more meaningful.

4.D. INTERNAL RATE OF RETURN

The primary strength of the internal rate of return (IRR) as a measure of the economic efficiency of an investment is that it accounts for the timeliness of all cash flows generated by that investment. The IRR of an investment is calculated by using a model similar to the present value series model discussed in Section 3.C:

$$PV = P_0 = \sum_{t=1}^{n} \frac{CF_t}{(1+r)^t}$$

or,

$$NPV = 0 = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t} \tag{4.6}$$

where net present value (NPV) is the present value of the series net of the initial cash outlay, and (r) is the return (or discount rate) that sets the investment's NPV equal to zero. The investment's internal rate of return is that value for (r) that equates NPV with zero.

There exists no general format allowing us to solve for the internal rate of return (r) in terms of the other variables in Equation (4.6); therefore, we must substitute values for (r) until we find one that works (unless a computer or calculator with a built-in algorithm for solving such problems can be accessed). Often, this substitution process is very time-consuming, but with experience calculating internal rates of return, one can find shortcuts to solutions in various types of problems. Perhaps, the most important shortcut will be to find an easy method for deriving an initial value to substitute for (r) resulting in an NPV fairly close to zero. One easy method for generating an initial value to substitute for (r) is by first calculating the investment's return on investment. If an investor wanted to calculate the internal rate of return for the first stock presented in Section 4.B, he may wish to first substitute for (r) the stock's 20% return on investment:

$$NPV = \frac{-100}{(1+r)^0} + \frac{10}{(1+r)^1} + \frac{10}{(1+r)^2} + \frac{10+130}{(1+r)^3}$$
$$= -100 + \frac{10}{(1+.2)^1} + \frac{10}{(1+.2)^2} + \frac{140}{(1+.2)^3} = -3.7$$

Since this NPV is less than zero, a smaller (r) value should be substituted. A smaller (r) value will decrease the right hand side denominators, increasing the size of the fractions and NPV. Perhaps a feasible value to substitute for (r) is 10%. The same calculations will be repeated with the new (r) value of 10%:

$$NPV = -100 + \frac{10}{1.1} + \frac{10}{(1.1)^2} + \frac{140}{(1.1)^3} = +\ 22.54$$

Since the new NPV exceeds zero, the (r) value of 10% is too small. However, because -3.70 is closer to zero than 22.54, the next value to substitute for (r) might be closer to 20% than to 10%. Perhaps a better estimate for the IRR will be 18%. Substituting this value for (r) results in an NPV of .86:

$$NPV = -100 + \frac{10}{1.18} + \frac{10}{(1.18)^2} + \frac{140}{(1.18)^3} = 0.86$$

This NPV is quite close to zero; in fact further substitutions will indicate that the true stock internal rate of return is approximately 18.369%. These iterations have a pattern: when NPV is less than zero, decrease (r) for the next substitution; when NPV exceeds zero, increase (r) for the next substitution. This process of iterations need only be repeated until the desired accuracy of calculations is reached. Figure 4.1 describes the relationship between the value selected for r and NPV.

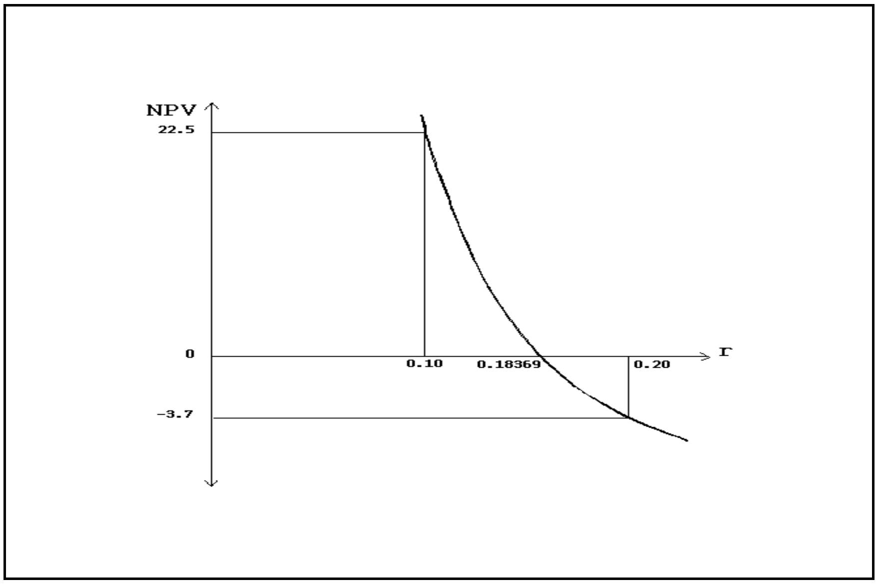

Figure 4.1: The relationship between NPV and r for stock one.

The primary advantage of the internal rate of return over return on investment is that it accounts for the timeliness of all cash flows generated by that investment. However, IRR does have three major weaknesses:

1. As we have seen, IRR takes considerably longer to calculate than does ROI. Therefore, if ease of calculation is of primary importance in a situation, the investor may prefer to use ROI as his measure of efficiency. As discussed in the appendix to this chapter, there are calculators and computer programs that will compute IRR very quickly.
2. Sometimes an investment will generate multiple rates of return; that is, more than one (r) value will equate NPV with zero. This will occur when that investment has associated with it more than one negative cash flow. When multiple rates are generated, there is often no method to determine which is the true IRR. In fact, none of the rates generated may make any sense. When the IRR is infeasible as a method for comparing two investments, and the investor still wishes to consider the time

value of money in his calculations, he may simply compare the present values of the investments.

3. The internal rate of return is based on the assumption that cash flows received prior to the expiration of the investment will be re-invested at the internal rate of return. That is, it is assumed that future investment rates are constant and equal to the IRR. Obviously, this assumption may not hold in reality.

4.E. BOND YIELDS

By convention, rates of return on bonds are often expressed in terms somewhat different from those of other investments. For example, the coupon rate of a bond is the annual interest payment associated with the bond divided by the bond's face value. Thus, a 4-year $1000 corporate bond making $60 annual interest payments has a coupon rate of 6%. However, the coupon rate does not account for the actual purchase price of the bond. Corporate bonds are usually traded at prices that differ from their face values. The bond's current yield accounts for the actual purchase price of the bond:

(4.7) $$cy = \frac{INT}{P_0}$$

If this bond were purchased for $800, its current yield would be 7.5%.

The formula for current yield, while easy to work with, does not account for any capital gains (or losses) that may be realized when the bond matures. Furthermore, current yields do not account for the timeliness of cash flows associated with bonds. The bond's yield to maturity, which is essentially its internal rate of return does account for any capital gains (or losses) that may be realized at maturity in addition to the timeliness of all associated cash flows:

(4.8) $$NPV = 0 = \sum_{t=0}^{n} \frac{CF_t}{(1+y)^t} = -P_0 + \sum_{t=1}^{n} \frac{INT}{(1+y)^t} + \frac{F}{(1+y)^n}$$

The yield to maturity (y) of the 4-year $1000 corporate bond above is 6% if it were purchased for $1,000; the yield to maturity would be 12.7% if the purchase price were $800. Regardless, (y) is identical to the bond's internal rate of return. If the bond makes semiannual interest payments, its yield to maturity can be more accurately expressed:

(4.9) $$NPV = 0 = \sum_{t=0}^{n} \frac{CF_t}{(1+y)^t} = -P_0 + \sum_{t=1}^{2n} \frac{INT/2}{(1+y/2)^t} + \frac{F}{(1+y)^n}$$

Here, we are concerned with semiannual interest payments and (2×n) 6-month time periods where (n) is the number of years to the bond's maturity. While yield to maturity is perhaps the most widely-used of the bond return measures, it still assumes a flat yield curve. This means that coupon payments received prior to bond maturity will be invested at the same rate as the bond's yield, an unrealistic assumption when interest rates are expected to change significantly over time.

4.F. INTRODUCTION TO RISK

When a firm invests, it subjects itself to at least some degree of uncertainty regarding future cash flows. Managers cannot know with certainty what investment payoffs will be. This chapter is concerned with forecasting investment payoffs and returns and the uncertainty associated with these forecasts. We will define expected return in this chapter, focusing on it as a return forecast. This expected return will be expressed as a function of the investment's potential return outcomes and associated probabilities. The riskiness of an investment is simply the potential for deviation from the investment's expected return. The risk of an investment is defined here as the uncertainty associated with returns on that investment. Although other definitions for risk such as the probability of losing money or going bankrupt can be very useful, they are often less complete or more difficult to measure. Our definition of risk does have some drawbacks as well. For example, an investment which is certain to be a complete loss is not regarded here to be risky since its return is known to be -100% (though we note that it probably would not be regarded to be a particularly good investment).

4.G. EXPECTED RETURN

Consider an economy with three potential states of nature in the next year and Stock A whose return is dependent on these states. If the economy performs well, state one is realized and the stock earns a return of 25%. If the economy performs only satisfactorily, state two is realized and the stock earns a return of 10%. If the economy performs poorly, state three is realized and the stock achieves a return of -10%. Assume that there is a 20% chance that state one will occur, a 50% percent chance that state two will occur and a 30% chance that state three will occur. The expected return on the stock will be 7%, determined by Equation (4.10):

$$E[R_A] = \sum_{i=1}^{n} R_{Ai} P_i \tag{4.10}$$

$$E[R_A] = (.25 \times .20) + (.10 \times .50) + (-.10 \times .30) = .07 ,$$

where (R_i) is return outcome (i) and (P_i) is the probability associated with that outcome. Therefore, our forecasted return is 7%. The expected return considers all potential returns and weights more heavily those returns that are more likely to actually occur. Although our forecasted return level is seven percent, it is obvious that there is potential for the actual return outcome to deviate from this figure. This potential for deviation (variation) will be measured in the following section.

4.H: VARIANCE AND STANDARD DEVIATION

The statistical concept of variance is a useful measure of risk. Variance accounts for the likelihood that the actual return outcome will vary from its expected value; furthermore, it accounts for the magnitude of the difference between potential return outcomes and the expected return. Variance can be computed with Equation (4.11):[1]

$$\sigma^2 = \sum_{i=1}^{n} (R_i - E[R])^2 P_i \qquad (4.11)$$

The variance of stock A returns presented in Section 4.G is .0156:

$$\sigma^2 = (.25-.07)^2 \times .2 \quad + (.10-.07)^2 \times .5 \quad + (-.10-.07)^2 \times .3 = .0156$$

The statistical concept of standard deviation is also a useful measure of risk. The standard deviation of a stock's returns is simply the square root of its variance:

$$\sigma = \sqrt{\sum_{i=1}^{n} (R_i - E[R])^2 P_i}$$

Thus, the standard deviation of returns on the stock described in Section 4.F is 12.49%.

Table 4.2: Expected Return, Variance and Standard Deviation of Returns for Stock A

i	R_i	P_i	R_iP_i	$R_i - E[R_a]$	$(R_i - E[R_a])^2$	$(R_i - E[R_a])^2P_i$
1	.25	.20	.05	.18	.0324	.00648
2	.10	.50	.05	.03	.0009	.00045
3	-.10	.30	-.03	-.17	.0289	.00867
		$E[R_a]$=.07				σ^2_a = .01560
						σ_a = .1249

[1]Readers without a background in statistics may wish to consult the statistics review in the on-line Elementary Mathematics Review.

Consider a second security, Stock B whose return outcomes are also dependent on economy outcomes one, two and three. If outcome one is realized, Stock B attains a return of 45%; in outcomes two and three, the stock attains returns of 5% and -15%, respectively. From Table 4.3, we see that the expected return on Stock B is seven percent, the same as for Stock A. However, the actual return outcome of Stock B is subject to more uncertainty. Stock B has the potential of receiving either a much higher or much lower actual return than does Stock A. For example, an investment in Stock B could lose as much as fifteen percent, whereas an equal investment in Stock A cannot lose more than ten percent. An investment in Stock B also has the potential of attaining a much higher return than an identical investment in Stock A. Therefore, returns on Stock B are subject to greater variability (or risk) than returns on Stock A. The concept of variance (or standard deviation) accounts for this increased variability. The variance of Stock B (.0436) exceeds that of Stock A (.0156), indicating that Stock B is riskier than Stock A.

Table 4.3: Expected Return, Variance, and Standard Deviation of Returns for Stock B

i	R_i	P_i	R_iP_i	$R_i - E[R_b]$	$(R_i - E[R_b])^2$	$(R_i - E[R_b])^2P_i$
1	.45	.20	.09	.38	.1444	.02888
2	.05	.50	.025	.02	.0004	.00020
3	-.15	.30	-.045	.22	.0484	.01452
		$E[R_b]$=.070				σ^2 =.04360
						σ_b = .2088

With the expected return and standard deviation of returns of an investment, we can establish ranges of potential returns and probabilities that actual returns will fall within these ranges if it appears that potential returns for that investment are normally distributed. For example, consider a third stock with normally distributed returns with an expected level of 7% and a standard deviation of 10%. From Table V in the text appendix, we see that there is a 68% probability that the actual return outcome on this stock will fall between -.03 and .17: (See the Statistics Review in the text appendix.)

$$E[R] - 1\sigma \;<\; R_i \;< E[R] + 1\sigma$$

$$(.07 - .10) < R_i < (.07 + .10)$$

A similar analysis indicates a 95% probability that the actual return outcome will fall between -.13 and .27:

$$E[R] - 2\sigma < R_i < E[R] + 2\sigma$$

$$(.07 - .20) < R_i < (.07 + .20)$$

Obviously, a smaller standard deviation of returns will lead to a narrower range of potential outcomes given any level of probability. If a security has a standard deviation of returns equal to zero, it has no risk. Such a security is referred to as the risk-free security with a return of (r_f). Therefore, the only potential return level of the risk-free security is (r_f). No such security exists in reality; however, short term United States treasury bills are quite close.The U.S. government has proven to be an extremely reliable debtor. When investors purchase treasury bills and hold them to maturity, they do receive their expected returns. Therefore, short-term treasury bills are probably the safest of all securities. For this reason, financial analysts often use the treasury bill rate (of return) as their estimate for (r_f) in many important calculations.

4.I: HISTORICAL VARIANCE AND STANDARD DEVIATION

Empirical evidence suggests that historical stock return variances (standard deviations) can be reasonable indicators of future variances (standard deviations). That is, a stock whose previous returns have been subject to substantial variability probably will continue to realize returns of a highly volatile nature. Therefore, past riskiness is often a good indicator of future riskiness. A stock's historical return variability can be measured with a historical variance:

$$\sigma_h^2 = \sum_{t=1}^{n} (R_t - \overline{R})^2 \frac{1}{n} \tag{4.12}$$

where (R_t) is the stock return in time (t) and ($\underline{R}$) is the historical average return over the (n) time period sample. The stock's historical standard deviation of returns is simply the square root of its variance. If an investor determines that the historical variance is a good indicator of its future variance, he may need not to calculate potential future returns and their associated probabilities for risk estimates; he may prefer to simply measure the stock's riskiness with its historical variance or standard deviation given as follows:

$$\sigma_H = \sqrt{\frac{\sum_{t=1}^{n}(R_i - \overline{R})^2}{n}}$$

Table 4.4 demonstrates historical variance and standard deviation computations for Stock D.

Table 4.4: Historical Variance and Standard Deviation of Returns of Stock D

t	R_t	$R_t - \overline{R}_D$	$(R_t - \overline{R}_D)^2$	$(R_t - \overline{R}_D)^2$ 1/n
1	.10	-.06	.0036	.00072
2	.15	-.01	.0001	.00002
3	.20	.04	.0016	.00032
4	.10	-.06	.0036	.00072
5	.25	.09	.0081	.00162
$\overline{R}_d$=.16				σ^2 =.00340
				σ_d =.05831

4.J. COVARIANCE

Standard deviation and variance provide us with measures of the absolute risk levels of securities. However, in many instances, it is useful to measure the risk of one security relative to the risk of another or relative to the market as a whole. The concept of covariance is integral to the development of relative risk measures. Covariance provides us with a measure of the relationship between the returns of two securities. That is, given that two securities returns are likely to vary, covariance indicates whether they will vary in the same direction or in opposite directions. The likelihood that two securities will comove in the same direction (or, more accurately, the strength of the relationship between returns on two securities) is measured by Equation (4.13):

$$\sigma_{k,j} = Cov[k, j] = \sum_{t=1}^{n}\left(R_{k,i} - E[R_k]\right)\left(R_{j,i} - E[R_j]\right)P_i \tag{4.13}$$

where (R_{ki}) and (R_{ji}) are the return of stocks (k) and (j) if outcome (i) is realized and (P_i) is the probability of outcome (i). E[R_k] and E[R_j] are simply the expected returns of securities (k) and (j). For example, the covariance between returns of stocks A and B is:

$$\text{cov}(A,B) = \{(.25 - .07) \times (.45 - .07) \times .20\}$$
$$+ \quad \{(.10 - .07) \times (.05 - .07) \times .50\} + \{(-.10 -.07) \times (-.15 -.07) \times .30\}$$

$$= \{.01368\} + \{-.0003\} + \{.01122\} = .0246.$$

Since this covariance is positive, the relationship between returns on these two securities is positive. That is, the larger the positive value of covariance, the more likely one security will perform well given that the second will perform well. A negative covariance indicates that strong performance by one security implies likely poor performance by the second security. A covariance of zero implies that there is no relationship between returns on the two securities. Table 4.5 details the solution method for this example.

Table 4.5: Covariance between Returns on Stocks A and B

i	R_{ai}	R_{bi}	P_i	$R_{ai}-E[R_a]$	$R_{bi}-E[R_b]$	$(R_{ai}-E[R_a])(R_{bi}-E[R_b])P_i$
1	.25	.45	.20	.18	.38	.01368
2	.10	.05	.50	.03	-.02	-.00030
3	-.10	-.15	.30	-.17	-.22	.01122
						COV(A,B) = .0246

Empirical evidence suggests that historical covariances are strong indicators of future covariance levels. Thus, if one is unable to associate probabilities with potential outcome levels, in many cases he may use historical covariance as his estimate for future covariance. Table 4.6 demonstrates how to determine historical covariance for two hypothetical stocks D and E.

Table 4.6: Historical Covariance between Returns on Stocks D and E

t	R_{dt}	R_{et}	$(R_{dt}-\bar{R}_d)$	$(R_{et}-\bar{R}_e)$	$(R_{dt}-\bar{R}_d)(R_{et}-\bar{R}_e)1/n$
1	.10	.15	-.06	-.05	.00060
2	.15	.18	-.01	-.02	.00004
3	.20	.25	.04	.05	.00040
4	.10	.20	-.06	0	0
5	.25	.22	.09	.02	.00036
	$\bar{R}_D$=.16	.20=$\bar{R}_E$			COV(D,E) = .00140

4.K: COEFFICIENT OF CORRELATION

The coefficient of correlation provides us with a means of standardizing the covariance between returns on two securities. For example, how large must covariance be to indicate a strong relationship between returns? Covariance will be smaller given low returns on the two securities than given high security returns. The coefficient of correlation (ρ_{kj}) between returns on two securities will always fall between -1 and +1.[1] If security returns are directly related, the correlation coefficient will be positive. If the two security returns always covary in the same direction by the same proportions, the coefficient of correlation will equal one. If the two security returns always covary in opposite directions by the same proportions, ($\rho_{k,j}$) will equal negative one. The stronger the inverse relationship between returns on the two securities, the closer ($\rho_{k,j}$) will be to negative one. If ($\rho_{k,j}$) equals zero, there is no relationship between returns on the two securities. The coefficient of correlation ($\rho_{k,j}$) between returns is simply the covariance between returns on the two securities divided by the product of their standard deviations:

(4.14) $$\rho_{kj} = \frac{COV(k, j)}{\sigma_k \cdot \sigma_j}$$

Equation (4.14) implies that the covariance formula can be rewritten:

(4.15) $$COV(k,j) = \sigma_k \sigma_j \rho_{k,j}$$

If an investor can access only raw data pertaining to security returns, he should first find security covariances then divide by the products of their standard deviations to find correlation coefficients. However, if for some reason the investor knows the correlation coefficients between returns on securities, he can use this value along with standard deviations to find covariances.

The coefficient of correlation between returns on stocks A and B is .96:

$$\rho_{kj} = \frac{.0246}{.1249 \cdot .21} = .96$$

This value can be squared to determine the coefficient of determination between returns of the two securities. The coefficient of determination ($\rho_{k,j}^2$) measures the proportion of variability in one security's returns that can be explained by or be associated with variability of returns on the second security. Thus, approximately 92% of the variability of stock A returns can be explained by or associated with variability of stock B returns. The concepts of covariance and correlation are

[1]Many statistics textbooks use the notation ($r_{i,j}$) to designate the correlation coefficient between variables (i) and (j). Because the letter (r) is used in this text to designate return, it will use the lower case rho (ρ_{ij}) to designate correlation coefficient.

crucial to the development of portfolio risk and relative risk models presented in later chapters.

Historical evidence suggests that covariances and correlations between stock returns remain relatively constant over time. Thus, an investor can use historical covariances and correlations as his forecasted values. However, it is important to realize that these historical relationships apply to standard deviations, variances, covariances and correlations, but not to the actual returns themselves. That is, we can often forecast future risk levels and relationships on the basis of historical data, but we cannot forecast returns on the basis of historical returns. Thus, last year's return for a given stock implies almost nothing about next year's return for that stock.

The historical covariance between returns on securities (i) and (j) can be found by solving Equation (4.16):

$$\sigma_{ij} = \sum_{t=1}^{n} (R_{i,t} - \overline{R}_i)(R_{J,t} - \overline{R}_j)\frac{1}{n} \tag{4.16}$$

where the sample data is taken from (n) years. See Table (4.6).

4.L: THE MARKET PORTFOLIO

As we shall see in the next chapter, a portfolio is simply a collection of investments. The market portfolio is the collection of all investments that are available to investors. That is, the market portfolio represents the combination or aggregation of all securities (or other assets) that are available for purchase. Investors may wish to consider the performance of this market portfolio to determine the performance of securities in general. Thus, the return on the market portfolio is representative of the return on the "typical" asset. An investor may wish to know the market portfolio return to determine the performance of a particular security or his entire investment portfolio relative to the performance of the market or a "typical" security.

Determination of the return on the market portfolio requires the calculation of returns on all of the assets available to investors. Because there are hundreds of thousands of assets available to investors (including stocks, bonds, options, bank accounts, real estate, etc.), determining the exact return of the market portfolio may be impossible. Thus, investors generally make use of indices such as the Dow Jones Industrial Average or the Standard and Poor's 500 (S&P 500) to gauge the performance of the market portfolio. These indices merely act as surrogates for the market portfolio; we assume that if the indices are increasing, then the market portfolio is performing well. For example, performance of the Dow Jones Industrials Average depends on the performance of the thirty stocks that comprise this index. Thus, if the Dow Jones market index is performing well, the thirty securities, on average are probably performing well. This strong performance may imply that the market portfolio is performing well. In any case, it is easier to measure the performance of a portfolio of thirty or five hundred stocks (for the Standard and Poor's 500) than it is to measure the performance of all of the securities that comprise the market portfolio.

4.M. CONCLUSION

Risk means different things to different people. To some, it means potential for not earning profits, for others it means potential for bankruptcy. In this text, we define risk to be uncertainty. There also exist numerous measures of risk. Among the more popular measures of risk is standard deviation. Standard deviation is a particularly useful indicator of risk for several reasons:

1. Standard deviation can accommodate all potential outcomes associated with an investment. Each outcome which varies significantly from the mean or expected value increases standard deviation.
2. Potential outcomes which are more likely affect standard deviations more than outcomes which are less likely. For example, a very likely outcome which deviates substantially from expected value will increase standard deviation.
3. The normal distribution, which describes the frequency of such a large number of financial phenomena, is defined in terms of standard deviation.

Historical standard deviation is useful when the analyst believes that the historical volatility of an investment is a good indicator of its future uncertainty.

Co-movement statistics such as covariance are quite useful in determining the relationship between various financial data. The sign of covariance indicates the direction of co-movement and is useful in measurement of portfolio risk and in the computation of relative risk statistics to be discussed later. Coefficient of correlation is essentially a standardized covariance; its absolute value indicates the intensity of co-movement. The coefficient of determination indicates the proportion of variability in one data set which can be explained by variability in a second data set.

Return and Risk

QUESTIONS AND PROBLEMS

4.1. An investor purchased one share of Drysdale stock for one hundred dollars in 2003 and sold it exactly one year later for $200. Calculate the investor's arithmetic mean return on investment.

4.2. An investor purchased one share of Wilson Company stock in 2006 for $20 and sold it in 2013 for $40. Calculate the following for this investor:

a. arithmetic mean return on investment.
b. geometric mean return on investment.
c. internal rate of return.

4.3. An investor purchased one hundred shares of Mathewson Company stock for $75 apiece in 1995 and sold each share for $80 exactly six years later. The Mathewson Company paid annual dividends of $8 per share in each of the six years the investor held the stock. Calculate the following for the investor:

a. arithmetic mean return on investment.
b. internal rate of return.

4.4. The Paige Baking Company is considering the purchase of an oven for $100,000 whose output will yield the company $20,000 in annual after-tax cash flows for each of the next five years. At the end of the fifth year, the oven will be sold for its $40,000 salvage value. Calculate the following for the machine Paige is considering purchasing:

a. arithmetic mean return on investment.
b. internal rate of return.

4.5. What is the net present value of an investment whose internal rate of return equals its discount rate?

4.6. An investor purchased one hundred shares each of Grove Company stock and Dean Company stock for $10 per share. The Grove Company paid an annual dividend of one dollar per share in each of the eight years the investor held the stock. The Dean Company paid an annual dividend of $.25 per share in each of the eight years the investor held the stock. At the end of the eight year period, the investor sold each of his shares of Grove Company stock for $11 and sold each of his shares of Dean Company stock for $18.

a. Calculate the sum of dividends received by the investor from each of the companies.
b. Calculate the capital gains realized on the sale of stock of each of the companies.
c. Calculate the return on investment for each of the two companies' stock using an arithmetic mean return.
d. Calculate the internal rate of return for each of the two stocks.
e. Which of the two stocks performed better during their holding periods?

4.7. The Lemon Company is considering the purchase of an investment for $100,000 that is expected to pay off $50,000 in two years, $75,000 in four years and $75,000 in six years. In the third year, Lemon must make an additional payment of $50,000 to sustain the investment. Calculate the following for the Lemon investment:

a. Return on investment using an arithmetic mean return.
b. The investment internal rate of return.
c. Describe any complications you encountered in part b.

4.8. A $1,000 face value bond is currently selling at a premium for $1,200. The coupon rate of this bond is 12% and it matures in three years. Calculate the following for this bond assuming its interest payments are made annually:

a. Its annual interest payments.
b. Its current yield.
c. Its yield to maturity.

4.9. Work through each of the calculations in Problem 4.8 assuming interest payments are made semi-annually.

4.10. The Nichols Company invested $100,000 into a small business twenty years ago. Its investment generated a cash flow equal to $3,000 in its first year of operation. Each subsequent year, the business generated a cash flow which was 10% larger than in the prior year; that is, the business generated a cash flow equal to $3,300 in the second year, $3,630 in the third year, and so on for nineteen years after the first. The Nichols Company sold the business for $500,000 after its twentieth year of operation. What was the internal rate of return for this investment?

4.11. Suppose that a mutual fund investing on behalf of shareholders yielded the following price and price performance results:

Date	t	P_t	P_{t-1}	DIV_t
June 30	1	50	-	0
July 31	2	55	50	0
Aug. 31	3	50	55	0
Sep. 30	4	54	50	0
Oct. 31	5	47	54	2
Nov. 30	6	51	47	0

Calculate for this fund monthly returns for each of the five months July to November and compute a geometric mean return over this 5-month period.

4.12. Mack Products management is considering the investment in one of two projects available to the company. The returns on the two projects (A) and (B) are dependent on the sales outcome of the company. Mack management has determined three potential sales outcomes (1), (2) and (3) for the company. The highest potential sales outcome for Mack is outcome (1) or $800,000. If this sales outcome were realized, Project (A) would realize a return outcome of 30%; Project (B) would realize a return of 20%. If outcome (2) were realized, the company's sales level would be \$500,000. In this case, project (A) would yield 15%, and Project (B) would yield 13%. The worst outcome (3) will result in a sales level of \$400,000, and return levels for Projects (A) and (B) of 1% and 9% respectively. If each sales outcome has an equal probability of occurring, determine the following for the Mack Company:

a. the probabilities of outcomes (1), (2) and (3).
b. its expected sales level.
c. the variance associated with potential sales levels.
d. the expected return of Project (A).
e. the variance of potential returns for Project (A).
f. the expected return and variance for Project (B).
g. standard deviations associated with company sales, returns on Project (A) and returns on Project (B).
h. the covariance between company sales and returns on Project (A).
i. the coefficient of correlation between company sales and returns on Project (A).
j. the coefficient of correlation between company sales and returns on Project (B).
k. the coefficient of determination between company sales and returns on Project (B).

4.13. Which of the projects in Problem 4.12 represents the better investment for Mack Products?

4.14. Historical <u>percentage</u> returns for the McCarthy and Alston Companies are listed in the following chart along with percentage returns on the market portfolio:

Year	McCarthy	Alston	Market
1988	4	19	15
1989	7	4	10
1990	11	-4	3
1991	4	21	12
1992	5	13	9

Calculate the following based on the preceding diagram:

a. mean historical returns for the two companies and the market portfolio.
b. variances associated with McCarthy Company returns and Alston Company returns as well as returns on the market portfolio.
c. the historical covariance and coefficient of correlation between returns of the two securities.
d. the historical covariance and coefficient of correlation between returns of the McCarthy Company and returns on the market portfolio.
e. the historical covariance and coefficient of correlation between returns of the Alston Company and returns on the market portfolio.

4.15. Forecast the following for both the McCarthy and Alston Companies based on your calculations in Problem 4.14:

a. variance and standard deviation of returns.
b. coefficient of correlation between each of the companies' returns and returns on the market portfolio.

4.16. The following table represents outcome numbers, probabilities and associated returns for stock A:

outcome (i)	return (R_i)	Probability (P_i)
1	.05	.10
2	.15	.10
3	.05	.05
4	.15	.10
5	.15	.10
6	.10	.10
7	.15	.10
8	.05	.10
9	.15	?
10	.10	.10

Thus, there are ten possible return outcomes for Stock A.
a. What is the probability associated with Outcome 9?
b. What is the standard deviation of returns associated with Stock A?

4.17. The Durocher Company management projects a return level of 15% for the upcoming year. Management is uncertain as to what the actual sales level will be; therefore, it associates a standard deviation of 10% with this sales level. Managers assume that sales will be normally distributed. What is the probability that the actual return level will:
a. fall between 5% and 25%?
b. fall between 15% and 25%?
c. exceed 25%?
d. exceed 30%?

4.18. What would be each of the probabilities in Problem 4.17 if Durocher Company management were certain enough of its forecast to associate a 5% standard deviation with its sales projection?

4.19. Under what circumstances can the coefficient of determination between returns on two securities be negative? How would you interpret a negative coefficient of determination? If there are no circumstances where the coefficient of determination can be negative, describe why.

4.20. Stock A will generate a return of 10% if and only if Stock B yields a return of 15%; Stock B will generate a return of 10% if and only if Stock A yields a return of 20%. There is a 50% probability that Stock A will generate a return of 10% and a 50% probability that it will yield 20%.
a. What is the standard deviation of returns for Stock A?

b. What is the covariance of returns between Stocks A and B?

4.21. An investor has the opportunity to purchase a risk-free treasury bill yielding a return of 10%. He also has the opportunity to purchase a stock which will yield either 7% or 17%. Either outcome is equally likely to occur. Compute the following:

a. the variance of returns on the stock.
b. the coefficient of correlation between returns on the stock and returns on the treasury bill.

4.22. The following daily prices were collected for each of three stocks over a twelve day period.

Company X		Company Y		Company Z	
DATE	PRICE	DATE	PRICE	DATE	PRICE
1/09	50.125	1/09	20.000	1/09	60.375
1/10	50.125	1/10	20.000	1/10	60.500
1/11	50.250	1/11	20.125	1/11	60.250
1/12	50.250	1/12	20.250	1/12	60.125
1/13	50.375	1/13	20.375	1/13	60.000
1/14	50.250	1/14	20.375	1/14	60.125
1/15	52.250	1/15	21.375	1/15	62.625
1/16	52.375	1/16	21.250	1/16	60.750
1/17	52.250	1/17	21.375	1/17	60.750
1/18	52.375	1/18	21.500	1/18	60.875
1/19	52.500	1/19	21.375	1/19	60.875
1/20	52.375	1/20	21.500	2/20	60.875

Based on the data given above, calculate the following:

a. Returns for each day on each of the three stocks. There should be a total of ten returns for each stock - beginning with the date 1/10.
b. Average daily returns for each of the three stocks.
c. Daily return standard deviations for each of the three stocks.

APPENDIX 4.A RETURN AND RISK SPREADSHEET APPLICATIONS

Table 4.A.1 contains spreadsheet entries for computing stock variances, standard deviations and covariances. The table lists daily closing prices for Stocks X, Y and Z from January 9 to January 20 in Cells B3:B14, E3:E14 and H3:H14. From these prices, we compute returns in Cells B19:B29, E19:E29 and H19:H29. Variance, standard deviation and covariance statistics in Rows 30 to 38 are computed from formulas displayed in Table 4.A.2.

Table 4.A.1: Stock Prices, Returns, Risk and Co-movement

	A	B	C	D	E	F	G	H
1	**CORP. X**				**CORP. Y**		**CORP.**	**Z**
2	**DATE**	**PRICE**		**DATE**	**PRICE**		**DATE**	**PRICE**
3	9-Jan	50.125		9-Jan	20		9-Jan	60.375
4	10-Jan	50.125		10-Jan	20		10-Jan	60.5
5	11-Jan	50.25		11-Jan	20.125		11-Jan	60.25
6	12-Jan	50.25		12-Jan	20.25		12-Jan	60.125
7	13-Jan	50.375		13-Jan	20.375		13-Jan	60
8	14-Jan	50.25		14-Jan	20.375		14-Jan	60.125
9	15-Jan	52.25		15-Jan	21.375		15-Jan	62.625
10	16-Jan	52.375		16-Jan	21.25		16-Jan	60.75
11	17-Jan	52.25		17-Jan	21.375		17-Jan	60.75
12	18-Jan	52.375		18-Jan	21.5		18-Jan	60.875
13	19-Jan	52.5		19-Jan	21.375		19-Jan	60.875
14	20-Jan	52.375		20-Jan	21.5		20-Jan	60.875
15								
16	**CORP. X**				**CORP. Y**		**CORP.**	**Z**
17	**DATE**	**RETURN**		**DATE**	**RETURN**		**DATE**	**RETURN**
18	9-Jan	N/A		9-Jan	N/A		9-Jan	N/A
19	10-Jan	0		10-Jan	0		10-Jan	0.00207
20	11-Jan	0.002494		11-Jan	0.00625		11-Jan	-0.00413
21	12-Jan	0		12-Jan	0.006211		12-Jan	-0.00207
22	13-Jan	0.002488		13-Jan	0.006173		13-Jan	-0.00208
23	14-Jan	-0.00248		14-Jan	0		14-Jan	0.002083
24	15-Jan	0.039801		15-Jan	0.04908		15-Jan	0.04158
25	16-Jan	0.002392		16-Jan	-0.00585		16-Jan	-0.02994
26	17-Jan	-0.00239		17-Jan	0.005882		17-Jan	0
27	18-Jan	0.002392		18-Jan	0.005848		18-Jan	0.002058
28	19-Jan	0.002387		19-Jan	-0.00581		19-Jan	0
29	20-Jan	-0.00238		20-Jan	0.005848		20-Jan	0
30	Mean	0.004064		Mean	0.006694		Mean	0.00087
31		0.000145		Variance	0.00022		Variance	0.000266

	A	B	C	D	E	F	G	H
	Variance							
32	Variance (P)	0.000132		Variance (P)	0.0002		Variance (P)	0.000241
33	St.D.	0.01204		St.D.	0.014842		St.D.	0.016296
34	St.D. (P)	0.011479		St.D. (P)	0.014151		St.D. (P)	0.015538
35			COV(X,Y)=	0.0001494		COV(Y,Z)=	0.000192	
36			COV(X,Z)=	0.000139				
37			CORR(X,Y)=	0.9196541		CORR(Y,Z)=	0.8733657	
38			CORR(X,Z)=	0.7791748				

Formulas for computing returns are given in Rows 19 to 29 in Table 4.A.2. Means, variances, standard deviations, covariances and correlation coefficients are computed in Rows 30 to 38. Row 30 computes the arithmetic mean return for each of the three stocks. Table 4.A.2 lists formulas associated with the values in cells A30:H38. The =(Average) function may be typed in directly as listed in Table 4.A.2 Row 30 or obtained from the Paste Function button (***f*$_x$**) menu under the Statistical sub-menu. Entry instructions are given in the dialogue box obtained when the Average function is selected. The variance formulas in Row 31 are based on the Sample formula; the Variance (P) formulas in Row 32 are based on the population formula. Standard deviation sample and population results are given in Rows 33 and 34. Covariances and correlation coefficients are given in Rows 35 to 38.

<u>Table 4.A.2: Stock Returns, Risk and Co-movement: Formula Entries</u>

	A	B	C	D	E	F	G	H
16	**CORP. X**			**CORP. Y**			**CORP. Z**	
17	**DATE**	**RETURN**		**DATE**	**RETURN**		**DATE**	**RETURN**
18	9-Jan	N/A		9-Jan	N/A		9-Jan	N/A
19	10-Jan	=B4/B3-1		10-Jan	=E4/E3-1		10-Jan	=H4/H3-1
20	11-Jan	=B5/B4-1		11-Jan	=E5/E4-1		11-Jan	=H5/H4-1
21	12-Jan	=B6/B5-1		12-Jan	=E6/E5-1		12-Jan	=H6/H5-1
22	13-Jan	=B7/B6-1		13-Jan	=E7/E6-1		13-Jan	=H7/H6-1
23	14-Jan	=B8/B7-1		14-Jan	=E8/E7-1		14-Jan	=H8/H7-1
24	15-Jan	=B9/B8-1		15-Jan	=E9/E8-1		15-Jan	=H9/H8-1
25	16-Jan	=B10/B9-1		16-Jan	=E10/E9-1		16-Jan	=H10/H9-1
26	17-Jan	=B11/B10-1		17-Jan	=E11/E10-1		17-Jan	=H11/H10-1
27	18-Jan	=B12/B11-1		18-Jan	=E12/E11-1		18-Jan	=H12/H11-1
28	19-Jan	=B13/B12-1		19-Jan	=E13/E12-1		19-Jan	=H13/H12-1
29	20-Jan	=B14/B13-1		20-Jan	=E14/E13-1		20-Jan	=H14/H13-1
30	Mean	=AVERAGE(B19:B29)		Mean	=AVERAGE(E19:E29)		Mean	=AVERAGE(H19:H29)
31	Variance	=VAR(B19:B29)		Variance	=VAR(E19:E29)		Variance	=VAR(H19:H29)
32	Variance (P)	=VARP(B19:B29)		Variance (P)	=VARP(E19:E29)		Variance (P)	=VARP(H19:H29)
33	St.D.	=STDEV(B19:B29)		St.D.	=STDEV(E19:E29)		St.D.	=STDEV(H19:H29)
34	St.D. (P)	=STDEVP(B19:B29)		St.D. (P)	=STDEVP(E19:E29)		St.D. (P)	=STDEVP(H19:H29)
35		COV(X,Y)=		=COVAR(B19:B29,E19:E29)	COV(Y,Z)=			=COVAR(E19:E29,H19:H29)
				=COVAR(B19:B29,H19:H29)				

36		COV(X,Z)=						
37		CORR(X,Y)=		=CORREL(B19:B29,E19:E29)	CORR(Y,Z)=			=CORREL(E19:E29,H19:H29)
38		CORR(X,Z)=		=CORREL(B19:B29,H19:H29)				