Chapter 3 Present Value and Securities Valuation

The objectives of this chapter are to enable you to:

- *Value cash flows to be paid in the future*
- *Value series of cash flows, including annuities and perpetuities*
- *Value growing annuities and perpetuities*
- *Value cash flows associated with stocks and bonds*
- *Understand how to amortize a loan*

3.A. INTRODUCTION

Cash flows realized at the present time have a greater value to investors than cash flows realized later for the following reasons:

1. *Inflation*: The purchasing power of money tends to decline over time.
2. *Risk*: One never knows with certainty whether he will actually realize the cash flow that he is expecting.
3. The option to either spend money now or defer spending it is likely to be worth more than being forced to defer spending the money.

The purpose of the Present Value concept is to provide a means of expressing the value of a future cash flow in terms of current cash flows. That is, the Present Value concept is used to determine how much an investor would pay now for the promise of some cash flow to be received at a later date. The present value of this cash flow would be a function of inflation, the length of wait before the cash flow is received, its riskiness and the time value an investor associates with money (how much he needs money now as opposed to later). Perhaps the easiest way to account for these factors when evaluating a future cash flow is to discount it in the following manner:

(3.1)
$$PV = \frac{CF_n}{(1+k)^n}$$

where (CF_n) is the cash flow to be received in year (n), (k) is an appropriate discount rate accounting for risk, inflation, and the investor's time value associated with money, and PV is the present value of that cash flow. The discount rate enables us to evaluate a future cash flow in terms of cash flows realized today. Thus, the maximum a rational investor would be willing to pay for an investment yielding a $9000 cash flow in six years assuming a discount rate of 15% would be $3891, determined as follows:

$$PV = \frac{\$9000}{(1+\cdot 15)^6} = \frac{\$9000}{2.31306} = \$3890.95$$

In the above example, we simply assumed a fifteen percent discount rate. Realistically, perhaps the easiest value to substitute for (k) is the current interest or return rate on loans or other investments of similar duration and riskiness. However, this market determined interest rate may not consider the individual investor's time preferences for money. Furthermore, the investor may find difficulty in locating a loan (or other investment) of similar duration and riskiness. For these reasons, more scientific methods for determining appropriate discount rates will be discussed later. In any case, the discount rate should account for inflation, the riskiness of the investment and the investor's time value for money.

3.B. DERIVING THE PRESENT VALUE FORMULA

The present value formula can be derived easily from the compound interest formula. Assume an investor wishes to deposit a sum of money into a savings account paying interest at a rate of fifteen percent, compounded annually. If the investor wishes to withdraw from his account $9,000 in six years, how much must he deposit now? This answer can be determined by solving the compound interest formula for X_0:

$$FV_n = X_0(1+i)^n; \quad X_0 = \frac{FV_n}{(1+i)^n} = \frac{\$9000}{(1+\cdot 15)^6} = \frac{\$9000}{2.31306} = \$3890.95$$

Therefore, the investor must deposit $3890.95 now in order to withdraw $9,000 in six years at fifteen percent.

Notice that the present value formula (3.1) is almost identical to the compound interest formula where we solve for the principal (X_0):

$$PV = \frac{CF_n}{(1+k)^n}; \quad X_0 = \frac{FV_n}{(1+i)^n}$$

Mathematically, these formulas are the same; however, there are some differences in their economic interpretations. In the interest formulas, interest rates are determined by market supply and demand conditions whereas discount rates are individually determined by investors themselves (although their calculations may be influenced by market interest rates). In the present value formula, we wish to determine how much some future cash flow is worth now; in the interest formula above, we wish to determine how much money must be deposited now to attain some given future value.

3.C. PRESENT VALUE OF A SERIES OF CASH FLOWS

If an investor wishes to evaluate a series of cash flows, he needs only to discount each separately and then sum the present values of each of the cash flows. Thus, the present value of a series of cash flows (CF_t) received in time period (t) can be determined by the following expression:[1]

(3.2)

$$PV = \sum_{t=1}^{n} \frac{CF_t}{(1+k)^t}$$

For example, if an investment were expected to yield annual cash flows of $200 for each of the next five years, assuming a discount rate of 5%, its present value would be $865.90:

$$PV = \frac{200}{(1+.05)^1} + \frac{200}{(1+.05)^2} + \frac{200}{(1+.05)^3} + \frac{200}{(1+.05)^4} + \frac{200}{(1+.05)^5} = 865.90$$

Therefore, the maximum price an individual should pay for this investment is $865.90 even though the cash flows yielded by the investment total $1000. Because the individual must wait up to five years before receiving the $1000, the investment is worth only $865.90. Use of the present value series formula does not require that cash flows (CF_t) in each year be identical, as does the annuity model presented in the next section.

3.D. ANNUITY MODELS

The expression for determining the present value of a series of cash flows can be quite cumbersome, particularly when the payments extend over a long period of time. This formula requires that (n) cash flows be discounted separately and then summed. When (n) is large, this task may be rather time-consuming. If the annual cash flows are identical and are to be discounted at the same rate, an annuity formula can be a useful time-saving device. The same problem discussed in the previous section can be solved using the following annuity formula:

(3.3)

$$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right]$$

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[1]Readers who are uncomfortable with the summation notation may wish to consult the mathematics appendix at the end of the text.

Derivation Box 3.1

Deriving the Present Value Annuity Equation

The present value annuity factor (pvaf) may be derived through use of the geometric expansion (See Chapter 2 Section F). Consider the case where we wish to determine the present value of an investment based on a cash flow of CF made at the end of each year t for n years where the appropriate discount rate is k:

(1) $$PVA = CF[(1 + k)^{-1} + (1 + k)^{-2} + ... + (1 + k)^{-n}]$$

Thus, the payment made at the end of the first year is discounted for one year, the payment at the end of the second year is discounted for two years, etc. Clearly, determining the present value of this account will be very time consuming if n is large. The first step of the geometric expansion is to multiply both sides of (1) by (1+k):

(2) $$PVA(1+k) = CF[1 + (1+k)^{-1} + ... + (1+k)^{-n+1}]$$

The second step in the geometric expansion is to subtract Equation (1) from Equation (2) to obtain:

(3) $$PV_A(1+k) - PV_A = CF[1 - (1+k)^{-n}] = CF\ [1 - 1/(1+k)^{n}]$$

which equals:

(4) $$PV_A \cdot 1 + PVA \times k - PVA = PVA \times k = CF\ [1 - 1/(1+k)^{n}]$$

Notice that the subtraction led to the cancellation of many terms, reducing the equation we wish to compute to a much more manageable size. Finally, we divide both sides of Equation (4) by k to obtain Equation (3.3):

(3.3) $$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right]$$

A more general discussion of the geometric expansion procedure is discussed in Appendix A at the end of Chapter 2.

where (CF) is the level of the annual cash flow generated by the annuity (or series). Use of this formula does require that all of the annual cash flows be identical. Thus, the present value of the cash flows in the problem discussed in the previous section is $865.90, determined as follows:

As (n) becomes larger, this formula becomes more useful relative to the present value series formula discussed in the previous section. However, the annuity formula requires that all cash flows be identical and be paid at the end of each year. The present value annuity formula can be derived easily from the perpetuity formula discussed in the next section or from the geometric expansion procedure described in the derivation box.

Note that each of the above calculations assumes that cash flows are paid at the end of each period. If, instead, cash flows were realized at the beginning of each period, the annuity would be referred to as an annuity due. Each cash flow generated by the annuity due would, in effect, be received one year earlier than if cash flows were realized at the end of each year. Hence, the present value of an annuity due is determined by simply multiplying the present value annuity formula by (1+k):

(3.4)
$$PV_{due} = \frac{CF}{K}\left[1 - \frac{1}{(1+k)^n}\right](1+k)$$

The present value of the five year annuity due discounted at five percent is determined:

$$PV_A = \frac{200}{(.05)}\left[1 - \frac{1}{(1+.05)^5}\right](1+.05) = 4000(.2164738)(1.05) = \$909.19$$

APPLICATION 3.1: Lotteries and Millionaires

In December 1996, newspapers around the U.S. announced the newly acquired millionaire status of forty three people who had participated in the purchase of the winning "Quik Pick" ticket in the Texas State Lottery. Each of the members of the winning coalition had paid ten dollars to purchase tickets and would receive a $1.083 million share in a prize totaling $46.6 million. Thirty nine of the forty three winners were residents in the West Texas town of Roby, a farming community of 616 which was struggling through the effects of an extended drought and several failed economic development schemes.

Just what does this new millionaire status mean to the lucky winners of the Quik Pick Lottery? First, several of the lottery winners were recipients of numerous unsolicited offers for investment advice and will have the opportunity to pay taxes on their winnings. More importantly, each winner will receive his $1.083 million in winnings in before-tax installments of $54,186 over a twenty year period. While the annual payments of $54,186 should prove most welcome to their recipients, what are these payments really worth? If we were to discount payments at an annual rate of 7%, we can compute their present value as follows:

$$PV = [\$54{,}186/.07]\times[1 - 1/(1+.07)^{20}] = \$574{,}047$$

Not only is the value of these payments substantially less than the $1.083 million payments to be made, at an annual interest rate of 7%, the $54,186 annual payment is not even as large as the interest on a million dollars. Furthermore, recipients must pay income taxes on their winnings. Thus, it seems that the Roby millionaires will probably not be able to enjoy the lavish lifestyles often associated with millionaires.

APPLICATION 3.2: Valuing Deferred Compensation

Many highly paid employees, including top managers at large firms and professional athletes have compensation packages which are paid over extended periods of time. Such arrangements are often mutually beneficial to both the employee and to the employer; the employee may realize benefits by deferring payment of taxes and the employer benefits from time value of money by deferring payments. Furthermore, the employer may benefit from improved performance resulting from the employee's the longer term focus. Both the employee and the employer should pay particular attention to the time value of money.

Consider the case of Alex Rodriguez, a star baseball player who, in December 2000, signed a ten-year contract with the Texas Rangers. The contract was reported to be worth over a quarter billion dollars. The deal called for Rodriguez to receive base salaries of $21 million each year from 2001 to 2004, $25 million in 2005 and 2006 and $27 million each year from 2007 to 2010. In addition, there would be annual $2 million installments on his signing bonus for each year from 2001 to 2005. These figures total $252 million. If we assume end of year payments to be discounted at 3%, ignore deferred compensation and accumulated interest, various performance incentives and certain options, we can calculate the present value of this compensation package in millions as follows:

$$PV = \mathbf{[}(\$21+2)/.03 \times (1-1/1.03^4)\mathbf{]} + \mathbf{[}(\$25+2)/(1.03^5)\mathbf{]} + \mathbf{[}\$25/(1.03^6)\mathbf{]}$$
$$+ \mathbf{[}((\$27)/.03 \times (1-1/1.03^4)]/(1.03^6))\mathbf{]} = 85.493 + 23.290 + 20.937 + 84.051$$
$$= \$213.772$$

Thus, on a discounted time value basis, Mr. Rodriguez’s base contract is only worth approximately $213,772,000. Fortunately, his contract contains provisions for significant increases starting in 2007 and for performance incentives.

Notice that the first four years of compensation are valued as a discounted annuity. Compensation for years five and six are valued as single discounted cash flows. Compensation for years seven through ten are valued as a seven-year annuity, deferred for one year after year six.

3.E. BOND VALUATION

Because the present value of a series of cash flows is simply the sum of the present values of the cash flows, the annuity formula can be combined with other present value formulas to evaluate investments. Consider, for example, a 7% coupon bond making annual interest payments for 9 years. If this bond has a $1,000 face (or par) value, and its cash flows are discounted at 6%, its value can be determined as follows:

$$PV = \frac{70}{.06}\left[1-\frac{1}{(1+.06)^9}\right]+\frac{1000}{(1+.06)^9} = \$476.118+591.898 = \$1068.017$$

Thus, the value of a bond is simply the sum of the present values of the cash flow streams resulting from interest payments and from principal repayment.

Now, let us revise the above example to value another 7% coupon bond. This bond will make semiannual (twice yearly) interest payments for 9 years. If this bond has a $1,000 face (or par) value, and its cash flows are discounted at the stated annual rate of 6%, its value can be determined as follows:

$$PV = \frac{35}{.03}\left[1-\frac{1}{(1+.03)^{18}}\right]+\frac{1000}{(1+.03)^{18}} = \$481.373+587.395 = \$1068.768$$

Again, the value of the bond is the sum of the present values of the cash flow streams resulting from interest payments and from the principal repayment. However, the semi-annual discount rate equals 3% and payments are made to bondholders in each of eighteen semi-annual periods.

3.F. PERPETUITY MODELS

As the value of (n) approaches infinity in the annuity formula, the value of the right hand side term in the brackets:

$$\frac{1}{(1+k)^n}$$

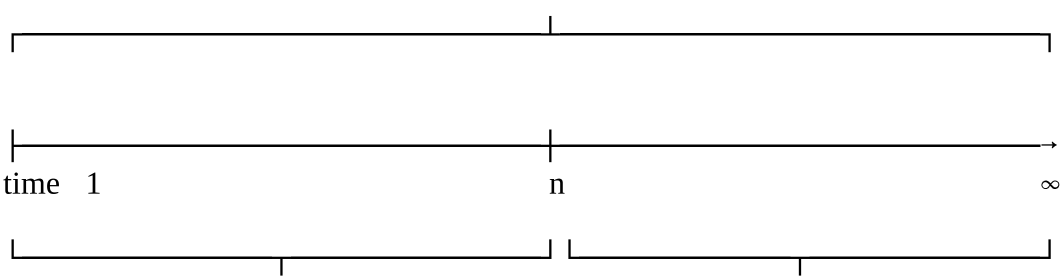
approaches zero. That is, the cash flows associated with the annuity are paid each year for a period approaching "forever." Therefore, as (n) approaches infinity, the value of the infinite time horizon annuity approaches:

(3.5) $$PV_p = \frac{CF}{k}$$

The annuity formula discussed in Section 3.D can be derived intuitively by use of Figure (3.1.a). First, consider a perpetuity as a series of cash flows beginning at time period one (one year from now) and extending indefinitely into perpetuity. Consider a second perpetuity with cash flows beginning in time period (n) and extending indefinitely into perpetuity. If an investor is to receive an (n) year annuity, the second perpetuity represents those cash flows from the first perpetuity that he will not receive. Thus, the difference between the present values of the first and second perpetuities represents the value of the annuity that he will receive. Note that the second perpetuity is discounted a second time since its cash flows do not begin until year (n):

$$PV_A = \frac{CF}{k} - \frac{\frac{CF}{k}}{(1+k)^n} = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right]$$

Present Value of Perpetuity Beginning in One Year = CF/k



Present Value of (n) Year Annuity: PVA

Present Value of Perpetuity Beginning in Year (n+1): $(CF/k)\div(1+k)^n$

The present value of a perpetuity beginning in one year minus the present value of a second perpetuity beginning in year (n+1) equals the present value of an (n) year annuity. Thus, PVA = $CF/k - (CF/k)\div(1+k^n) = CF/k \cdot [1 - 1/(1+k^n)]$

Figure 3.1.a: Deriving Annuity Present Value from Perpetuity Present Values

The perpetuity model is useful in the evaluation of a number of investments. Any investment with an indefinite or perpetual life expectancy can be evaluated with the perpetuity model. For example, the present value of a stock, if its dividend payments are projected to be stable, will be equal to the amount of the annual dividend (cash flow) generated by the stock divided by an appropriate discount rate. In European financial markets, a number of perpetual bonds have been traded for several centuries. In many regions in the United States, ground rents (perpetual leases on land) are traded. The proper evaluation of these and many other investments requires the use of perpetuity models.

The maximum price an investor would be willing to pay for a perpetual bond generating an annual cash flow of \$200, each discounted at a rate of 5% can be determined from Equation (3.5):

$$PV_p = \frac{\$200}{.05} = \$4000$$

3.G. GROWING PERPETUITY AND ANNUITY MODELS

If the cash flow associated with an investment were expected to grow at a constant annual rate of (g), the amount of the cash flow generated by that investment in year (t) would be:

$$CF_t = CF_1(1+g)^{t-1} \quad , \tag{3.6}$$

where (CF_1) is the cash flow generated by the investment in year one. Thus, if a stock paying a dividend of \$100 in year one were expected to increase its dividend payment by 10% each year thereafter, the dividend payment in the fourth year would be \$133.10:

$$CF_4 = CF_1\,(1 + .10)^{4-1}$$

Similarly, the cash flow generated by the investment in the following year (t+1) will be:

$$CF_{t+1} = CF_1\,(1 + g)^t \tag{3.7}$$

The stock's dividend in the fifth year will be \$146.41:

$$CF_{4+1} = CF_1\,(1+.10)^4 = \$146.41$$

If the stock had an infinite life expectancy (as most stocks might be expected to), and its dividend payments were discounted at a rate of 13%, the value of the stock would be determined by:

$$PV_{gp} = \frac{\$100}{.13-.10} = \frac{\$100}{.03} = \$3333.33$$

This expression is called the Gordon Stock Pricing Model. It assumes that the cash flows (dividends) associated with the stock are known in the first period and will grow at a constant compound rate in subsequent periods. More generally, this growing perpetuity expression can be written as follows:

(3.8)
$$PV_{gp} = \frac{CF_1}{k-g}$$

The growing perpetuity expression simply subtracts the growth rate from the discount rate; the growth in cash flows helps to "cover" the time value of money. This formula for evaluating growing perpetuities can be used only when (k) > (g). If (g) > (k), either the growth rate or discount rate has probably been calculated improperly. Otherwise, the investment would have an infinite value (even though the formula would generate a negative value).

The formula (3.8) for evaluating growing annuities can be derived intuitively from the growing perpetuity model. In Figure (3.1.b), the difference between the present value of a growing perpetuity with cash flows beginning in time period (n) is deducted from the present value of a perpetuity with cash flows beginning in year one, resulting in the present value of an (n) year growing annuity. Notice that the amount of the cash flow generated by the growing annuity in year (n+1) is $CF(1+g)^n$. This is the first of the cash flows not generated by the growing annuity; it is generated after the annuity is sold or terminated. Because the cash flow is growing at the rate (g), the initial amount of the cash flow generated by the second perpetuity is exceeded by the initial cash flow of the perpetuity beginning in year one.

(3.9)
$$PV_{GA} = \frac{CF_1}{k-g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n}\right]$$

APPLICATION 3.3: Valuing the Lifetime Membership

In December 1995, the *Journal of Finance* offered a special holiday rate for a lifetime individual membership to the American Finance Association for $1110. Dues at that time were $63 per year. Would a lifetime membership to an individual expecting to be active professionally for 35 years have been a good purchase? Consider the following computations based on an assumed discount rate of 8% and an assumed growth rate in the annual membership fee equal to 3%:

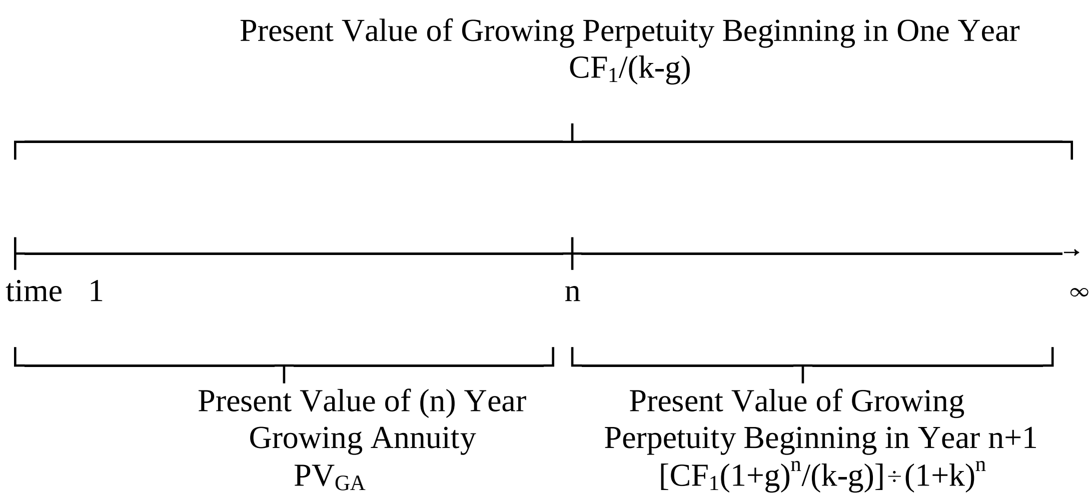
$$PV_{ga} = [CF_1/(k\text{-}g)] \times [1 - (1+g)^n /(1+k)^n]$$
$$= [\$63/(.08\text{-}.03)] \times [1 - (1+.03)^{35}/(1+.08)^{35}] = 1020.20$$

Note that we are only valuing the membership for the 35 year period the individual expects to remain professionally active. Thus, because the present value of anticipated annual dues is less than the $1110 lifetime membership cost, the lifetime membership seems to be a reasonable expenditure.

Cash flows generated by many investments will grow at the rate of inflation. For example, consider a project undertaken by a corporation whose cash flow in year one is expected to be \$10,000. If cash flows were expected to grow at the inflation rate of six percent each year until year six, then terminate, the project's present value would be \$48,320.35, assuming a discount rate of 11%:

$$PV_{GA} = \frac{\$10,000}{.11-.06} \cdot \left[1 - \frac{(1+.06)^6}{(1+.11)^6} \right] = \$200,000(1-.7584) = \$48320.45$$

Note that we have calculated the present value of a growing annuity, whose cash grew for a finite period and then terminated. Cash flows are generated by this investment through the end of the sixth year. No cash flow was generated in the seventh year. Verify that the amount of cash flow that would have been generated by the investment in the seventh year if it had continued to grow would have been $\$10,000(1.06)^6 = \$14,185$.



The present value of a growing perpetuity beginning in one year minus the present value of a second growing perpetuity beginning in year (n+1) equals the present value of an (n) year growing annuity: $CF_1/(k\text{-}g) - [CF_1/(k\text{-}g)] \div (1+k)^n$; $PV_{GA} = [CF_1/(k\text{-}g)][1-(1+g)]^n \div (1+k)^n$

Figure 3.1.b: **Deriving Growing Annuity Present Value from Growing Perpetuity Present Value**

3.H. STOCK VALUATION

Consider a stock whose annual dividend next year is projected to be \$50. This payment is expected to grow at an annual rate of 5% in subsequent years. An investor has determined that the appropriate discount rate for this stock is 10%. The current value of this stock is \$1000, determined by the growing perpetuity model:

$$PV_{gp} = \frac{\$50}{.10-.05} = \$1000$$

This model is often referred to as the Gordon Stock Pricing Model. It may seem that this model assumes that the stock will be held by the investor forever. But what if the investor intends to sell the stock in five years? Its value would be determined by the sum of the present values of cash flows the investor does expect to receive:

$$PV_{GA} = \frac{DIV_1}{k-g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n}\right]$$

where (P_n) is the price the investor expects to receive when he sells the stock in year (n); and (DIV_1) is the dividend payment the investor expects to receive in year one. The present value of the dividends the investor expects to receive is \$207.53:

$$PV_{GA} = \frac{\$50}{.10-.05} \cdot \left[1 - \frac{(1+.05)^5}{(1+.10)^5}\right] = 207.53$$

The selling price of the stock in year five will be a function of the dividend payments the prospective purchaser expects to receive beginning in year six. Thus, in year five, the prospective purchaser will pay \$1,276.28 for the stock, based on his initial dividend payment of \$63.81, determined by the following equations:

$$DIV_6 = DIV_1\ (1+.05)^{6-1} = \$63.81$$

$$\text{Stock value in year five} = 63.81/(.10-.05) = \$1276.28$$

The present value of the \$1,276.28 the investor will receive when he sells the stock at the end of the fifth year is \$792.47:

$$PV = \frac{\$1276.28}{(1+.1)^5} = \$792.57$$

The total stock value will be the sum of the present values of the dividends received by the investor and his cash flows received from the sale of the stock. Thus, the current value of the stock is \$207.53 plus \$792.47, or \$1000. This is exactly the same sum determined by the growing perpetuity model earlier; therefore, the growing perpetuity model can be used to evaluate a stock even when the investor expects to sell it.

3.I. AMORTIZATION

In the beginning of this chapter, we derived the concept of present value from that of future value. Amortization is essentially a topic relating to interest, but the present value annuity model presented in this chapter is crucial to its development. Amortization is the payment structure associated with a loan. That is, the amortization schedule of a loan is its payment schedule. Consider the annuity model (3.3):

$$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right] \tag{3.3}$$

Typically, when a loan is amortized, the loan repayments will be made in equal amounts; that is, each annual or monthly payment will be identical. At the end of the repayment period, the balance (amount of principal remaining) on the loan will be zero. Thus, each payment made by the borrower is applied to the principal repayment as well as to interest. A bank loaning money will require that the sum of the present values of its repayments be at least as large as the sum of money it loans. Therefore, if the bank loans a sum of money equal to (PV) for (n) years at an interest rate of (i), the amount of the annual loan repayment will be (CF):

$$CF = PV_A k \div \left[1 - \frac{1}{(1+k)^n}\right] \tag{3.10}$$

For example, if a bank were to extend a \$865,895 five year mortgage to a corporation at an interest rate of 5%, the corporation's annual payment on the mortgage would be \$200,000, determined by Equation (3.10):

$$CF = [\$865{,}895 \times .05] \div \left[1 - \frac{1}{(1+.05)^5}\right] = \$200{,}000$$

Thus, each year, the corporation will pay \$200,000 towards both the loan principal and interest obligations. The amounts attributed to each are given in Table 3.1. Notice that as payments are applied toward the principal, the principal declines; correspondingly, the interest payments decline. Nonetheless, total annual payments are identical until the principal diminishes to zero in the fifth year.

Year	Principal	Payment	Interest	Payment to Principal
1	865,895	200,000	43,295	156,705
2	709,189	200,000	35,459	164,541
3	544,649	200,000	27,232	172,768
4	371,881	200,000	18,594	181,406
5	190,476	200,000	9,524	190,476

Note: The loan is fully repaid by the end of the fifth year. The principal represents the balance at the beginning of the given year. The payment is made at the end of the given year, and includes one year of interest accruing on the principal from the beginning of that year. The remaining part of the payment is payment to the principal. This payment to the principal is deducted from the principal or balance as of the beginning of the following year.

TABLE 3.1: Amortization schedule of $865,895 loan with equal annual payments for five years at 5% interest

Consider a second example where a family is considering the purchase of a home with $50,000 down and a $500,000 mortgage. The mortgage will be amortized over thirty years with equal monthly payments. The interest rate on the mortgage will be 8% per year. Based on this data, we would like to determine the monthly mortgage payment and compile an amortization table decomposing each of the monthly payments into interest and payment toward principle.

First, we will express annual data as monthly data. Three hundred sixty (12 × 30) months will elapse before the mortgage is fully paid and the monthly interest rate will be .00667 or 8% divided by 12. Given this monthly data, monthly mortgage payments are determined as follows:

$$Payment = [\$500{,}000 \times .00667] \div \left[1 - \frac{1}{(1+.00667)^{360}}\right] = \$3{,}670.22$$

Table 3.2 depicts the several rows taken from the amortization schedule for this mortgage.

Month	Beginning of Month Principal	Total Payment	Payment on Interest	Payment on Principal
1	500,000.00	3,670.22	3,335.00	335.22
2	499,664.80	3,670.22	3,332.76	337.45
3	499,327.30	3,670.22	3,330.51	339.70
4	498,987.60	3,670.22	3,328.28	341.97
5	498,645.70	3,670.22	3,325.97	344.25
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
358	10,865.39	3,670.22	72.47	3,597.75
359	7,267.64	3,670.22	48.48	3,621.74
360	3,645.90	3,670.22	24.32	3,645.90

TABLE 3.2: Amortization schedule of $500,000 loan with equal annual payments for 360 months at .667% interest

3.J. CONCLUSION

This chapter has demonstrated how to value anticipated cash flows based on the discounting technique. Single cash flows are evaluated, as are cash flow series, annuities and perpetuities. Constant growth models were evaluated and the models were applied to the evaluation of bonds, preferred stock and common stock. One of the most useful valuation expressions is the growing annuity formula (3.9):

$$PV_{GA} = \frac{CF_1}{k-g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n} \right] \tag{3.9}$$

This expression is quite general in that it can be applied to a variety of circumstances. For example, the term n can be set to ∞ in the case of a growing perpetuity such that the expression in the brackets falls to zero. The term g can be set to zero for valuing no-growth annuities. However, this expression cannot be used when the cash flow streams are uneven or do not grow at a constant compound rate. In this case, the more general present value series expression (3.2) must be used:

$$PV = \sum_{t=1}^{n} \frac{CF_t}{(1+k)^t}$$

This formula is sufficiently general to accommodate any series of cash flows, constant, varying or growing at either constant or varying intervals. On the other hand, its computation can be quite time consuming when n is large.

QUESTIONS AND PROBLEMS

3.1. What is the present value of a security promising to pay $10,000 in five years if its associated discount rate is:

a. twenty percent?
b. ten percent?
c. one percent?
d. zero percent?

3.2. What is the present value of a security to be discounted at a ten percent rate promising to pay $10,000 in:

a. twenty years?
b. ten years?
c. one year?
d. six months?
e. seventy three days?

3.3. The Gehrig Company is considering an investment that will result in a $2000 cash flow in one year, a $3000 cash flow in two years and a $7000 cash flow in three years. What is the present value of this investment if all cash flows are to be discounted at an eight percent rate? Should Gehrig Company management be willing to pay $10,000 for this investment?

3.4. The Hornsby Company has the opportunity to pay $10,000 for an investment paying $2000 in each of the next nine years. Would this be a wise investment if the appropriate discount rate were:

a. five percent?
b. ten percent?
c. twenty percent?

3.5. The Foxx Company is selling preferred stock which is expected to pay a fifty dollar annual dividend per share. What is the present value of dividends associated with each share of stock if the appropriate discount rate were eight percent and its life expectancy were infinite?

3.6. The Evers Company is considering the purchase of a machine whose output will result in a ten thousand dollar cash flow next year. This cash flow is projected to grow at the annual ten percent rate of inflation over each of the next ten years. What will be the cash flow generated by this machine in:

a. its second year of operation?
b. its third year of operation?
c. its fifth year of operation?
d. its tenth year of operation?

3.8. The Wagner Company is considering the purchase of an asset that will result in a $5000 cash flow in its first year of operation. Annual cash flows are projected to grow at the 10% annual rate of inflation in subsequent years. The life expectancy of this asset is seven years, and the appropriate discount rate for all cash flows is twelve percent. What is the maximum price Wagner should be willing to pay for this asset?

3.8. What is the present value of a stock whose $100 dividend payment next year is projected to grow at an annual rate of five percent? Assume an infinite life expectancy and a twelve percent discount rate.

3.9. Which of the following series of cash flows has the highest present value at a five percent discount rate:

a. $500,000 now
b. $100,000 per year for eight years
c. $60,000 per year for twenty years
d. $30,000 each year forever

3.10. Which of the cash flow series in Problem 3.9 has the highest present value at a twenty percent discount rate?

3.11. Mr. Sisler has purchased a $200,000 home with $50,000 down and a twenty year mortgage at a ten percent interest rate. What will be the periodic payment on this mortgage if they are made:

a. annually?
b. monthly?

3.12. What discount rate in Problem 3.4 will render the Hornsby Company indifferent as to its decision to invest $10,000 for the nine year series of cash flows? That is, what discount rate will result in a $10,000 present value for the series?

3.13. What would be the present value of $10,000 to be received in twenty years if the appropriate discount rate of 10% were compounded:

a. annually?
b. monthly?
c. daily?
d. continuously?

3.14.a. What would be the present value of a thirty year annuity if the $1000 periodic cash flow were paid monthly? Assume a discount rate of 10% per year.

b. Should an investor be willing to pay $100,000 for this annuity?

c. What would be the highest applicable discount rate for an investor to be willing to pay $100,000 for this annuity?

3.15. Demonstrate how to derive an expression to determine the present value of a growing annuity. Use the geometric expansion to derive Equation (3.12) in the text.

3.16. An individual has purchased a home with $30,000 down and a $300,000 mortgage. The mortgage will be amortized over thirty years with equal monthly payments. The interest rate on the mortgage will be 9% per year. Based on this data, answer the following:

a. How many months will elapse before the mortgage is fully paid?
b. What is the monthly interest rate on the mortgage?
c. What will be the monthly mortgage payment?
d. Set up an amortization table to illustrate interest payments, payments on the principal and mortgage balances (beginning of month principal).

3.17.* Suppose an investor has the opportunity to invest in a stock currently selling for $100 per share. The stock is expected to pay a $5 dividend next year (at the end of year 1). In each subsequent year through the third year, the annual dividend is expected to grow at a rate of 15%. Starting in the fourth year, the annual dividend will grow at an annual rate of 6% until the sixth year. Starting in the seventh year, dividends will not grow, but will remain the same as in the sixth year. All cash flows are to be discounted at an annual rate of 8%. Should the stock be purchased at its current price of $100?

APPENDIX

3.A. TIME VALUE SPREADSHEET APPLICATIONS

Spreadsheets are very useful for time value calculations, particularly when there are either a large number of time periods or a large number of potential outcomes. Not are most time value formulas easy to enter into cells, but the toolbar the top of the Excel screen should have the Paste Function button (***f*x**) which will direct the user to a variety of time value functions. By left-clicking the Paste Function (***f*x**), the user will be directed to the Paste Function menu. From the Paste Function menu, one can select the Financial sub-menu. In the Financial sub-menu, scroll down to select the appropriate time value function. Pay close attention to the proper format and arguments for entry. Table 3.B.1 below lists a number of time value functions which may be accessed through the Paste Function menu along with the example and notes.

While the formulas entered into Table 3.B.1 make use of specialized Paste Functions for Finance, the spreadsheet user can enter his own simple formulas. For example, suppose that the user enters a cash flow in cell A1, a discount rate in cell A2 and a payment or termination period into cell A3. The present value of this cash flow can be found with =A1/(1+A2)^A3 or, in the case of an annuity, with =A1*((1/A2)-(1/(A2*(1+A2)^A3))). Now, enter a deposit amount into cell A1, an interest rate in cell A2 and a payment date or termination date in cell A3. Future values can be found with =A1*(1+A2)^n and =A1*((1/A2)-(1/(A2*(1+A2)^A3)))*(1+A2)^n. These formulas can easily be adjusted for growth, in which a value for cell A4 may be inserted for the growth rate.

Table A.3.1: Time Value Formula Entry and Paste Functions

	=Function Type from *fx*	Format	Entry Example	Result	Formula Entry
1	Future Value of Single Deposit or Investment			-1610.51	=-1000*(1+0.1)^5
2	Future Value of Annuity	=FV(i, n, CF)	=PV(.1,5,100)	-610.51	=-100*((1+0.1)^5-1)/0.1
3	Future Value of Annuity with FV	=FV(i, n, CF, FV, Type)	=PV(.1,5,100,1000,0)	-2221.02	=-100*((1+0.1)^5-1)/0.1-1000*(1+0.1)^5
4	Future Value of Annuity Due with FV	=FV(i, n, CF, FV, Type)	=PV(.1,5,100,1000,1)	-2282.07	=-100*((1+.1)^6-(1+.1))/.1-1000*(1+.1)^5
5	Present Value of Future Cash Flow			-620.92	=-1000/(1+0.1)^5
6	Net Present Value of Series	=NPV(k,Value 0,Value 1,Value 2,etc.)	=NPV(0.1,-100,110)	0.00	=100/(1+0.1)^0-110/(1+0.1)^1
7	Present Value of Annuity	=PV(k, n, CF)	=PV(.1,5,100)	-379.08	=-100/0.1*(1-1/(1+0.1)^5)
8	Present Value of Annuity with FV	=PV(k, n, CF, FV, Type)	=PV(.1,5,100,1000,0)	-1000.00	=-100/0.1*(1-1/(1+0.1)^5)-1000/(1+0.1)^5
9	Present Value of Annuity Due with FV	=PV(k, n, CF, FV, Type)	=PV(.1,5,100,1000,1)	-1037.91	=-100/.1*(1-1/(1+.1)^5)*(1+.1)-1000/(1+.1)^5
10	Amortized Payment on Loan	=PMT(I,n,PV)	=PMT(0.1,5,1000)	-263.80	=-1000*0.1/(1-1/(1+0.1)^5)
11	Amortized Payment on Loan (Due)	=PMT(I,n,PV,0,Type)	=PMT(0.1,5,1000,1000,1)	-239.82	=(-1000*0.1)/((1-1/(1+0.1)^5)*(1+0.1))

Notes

For all the functions above (except 1 and 5), one can either use the *fx* format or the Paste Function Menu retrievable from the toolbar above the spreadsheet.

For all the functions, the formula entry method from the right is usable. See notes below on individual function use. Also, notice how the spreadsheet interprets negative and positive cash flows.

1 Just enter Formula from the right to find the future value of a single cash flow. In this example, X = $1000, i = 10% and n = 5.

2 The example is for a $100 annuity for 5 years at 10%.

3 The example is for a $100 annuity for 5 years at 10% plus an additional $1000 deposited at time 0.

4 Same as 3 but with all cash flows at the beginning of the periods.

5 Just enter Formula from the right to find the present value of a single cash flow. In this example, CF = $1000, k = 10% and n=5.

6 Net Present value of a series of cash flows starting at time 0. Just enter the amounts of the cash flows which can vary from year-to-year.

7 The example is for a $100 annuity for 5 years at 10%.

8 The example is for a $100 annuity for 5 years at 10% plus $1000 received at time 5.

9 Same as 9 but with beginning-of-year cash flows.

10 Amortized (mortgage) payment on a $1000 loan at 10% for 5 years.

11 Same as 10 but with beginning-of-year payments.