Chapter 2   Interest and Future Value

The objectives of this chapter are to enable you to:

- *Understand the relationship between interest and future value*
- *Calculate future values based on single investments*
- *Compare investments with different compounding intervals*
- *Calculate future values based on multiple investments*
- *Understand annuity future value formulas*

2.A: INTRODUCTION

The expression “Time is money” certainly applies in finance. People and institutions are impatient; they want money now and are generally willing to pay (or impose a charge) for having money now (or having to wait). The time value of money is certainly among the most important concepts in finance.

Interest is a charge imposed on borrowers for the use of lenders' money. The interest cost is usually expressed as a percentage of the *principal* (the sum borrowed). When a loan matures, the principal must be repaid along with any unpaid accumulated interest.

In a free market economy, interest rates are determined jointly by the supply of and demand for money. Thus, lenders will usually attempt to impose as high an interest rate as possible on the money they lend; borrowers will attempt to obtain the use of money at the lowest interest rates available to them. Factors affecting the levels of interest rates will do so by affecting supply and demand conditions for money. Among these factors are:

1. *Inflation*: Because of diminished purchasing power, money received in the future by lenders is worth less than the money they lend now. Lenders will require a premium (interest) in addition to the principal to compensate them for this loss of purchase power. Furthermore, inflation makes current money balances more attractive to borrowers. Thus, inflation decreases the supply of and increases the demand for money. Interest rates will increase as the rate of inflation increases. (See Figures 2.1.a and 2.1.b)
2. *Risk or Uncertainty*: Creditors naturally prefer to know with certainty that the money they loan will be repaid in its entirety. If lenders are uncertain as to whether their loans will be repaid, they will require premiums to compensate them for this risk. Higher interest rates will result from increased uncertainty.
3. *Intertemporal Monetary Preferences*: In general, consumers (and corporations) will prefer to have money now rather than be forced to wait for it. If consumers have money now, they can choose to spend it now or spend it at some later date.

However, if consumers must wait for their money, they do not have the option to spend it now; they must wait for some later date to spend it. If consumers increase their desire to spend more now rather than later, interest rates will increase.

4. *Government Policy*: Governmental monetary policy will affect both supply and demand conditions for money. Through monetary policy, the government can directly control the supply of money; and through its participation in bond markets, it can influence the demand for money. Governmental fiscal policy (spending and tax programs) have a significant effect on the demand for money.
5. *Costs of Extending Credit*: Both lenders and borrowers face various negotiating and administrative costs when a loan is extended. Most of these costs can be categorized as transactions costs. Lenders will require initiation fees such as "points" or higher interest payments as compensation for these costs.

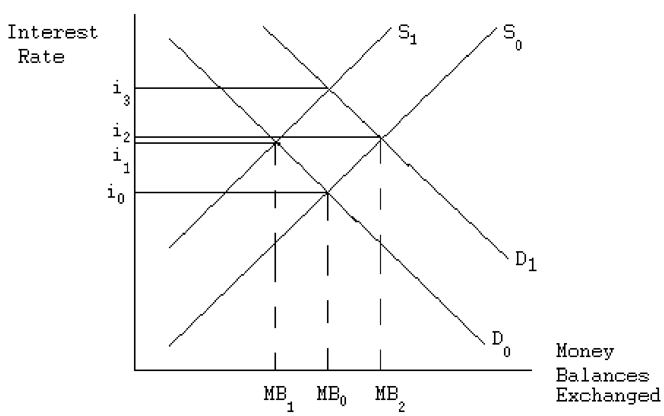

Figure 2.1.a: Factors Increasing Interest Rates

Consider each of the following scenarios that start from initial interest rate (i_0) and money balance (MB_0):

1. Increased lending risk cause suppliers of capital to be less willing to lend. The supply curve for money balances shifts back to (S_1) and the interest rate rises to (i_1).
2. Expansionary fiscal policy increases the demand for money balances to (D_1) and increases the interest rate to (i_2).
3. Increased inflation increases the demand for monetary balances while decreasing lenders' willingness to supply capital. The demand curve for balances shifts out to (D_1); the supply curve shifts back to (S_1). Interest rates rise to (i_3).
4. Increased negotiating and administrative costs incurred by lenders decrease the amount of credit they are willing to extend. Interest rates rise to (i_1).

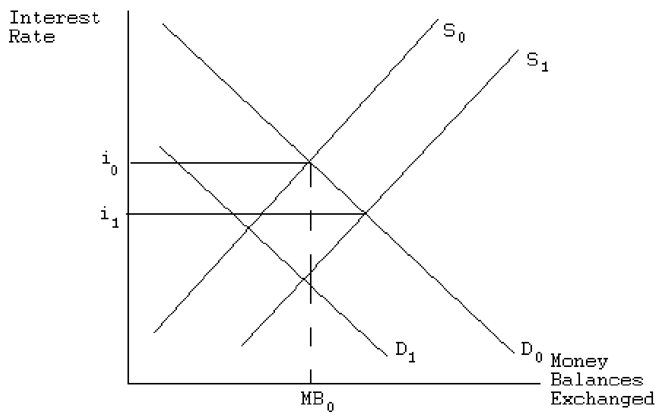

Figure 2.1.b: Factors Decreasing Interest Rates

Consider each of the following scenarios that start from initial interest rate (i_0) and money balance (MB_0):

1. Expansionary monetary policy increases the supply of money balances to (S_1), causing interest rates to decline to (i_1)
2. Alternatively, increased consumer willingness to save decreases the demand for capital to (D_1), resulting in interest rates declining to (i_1)

2.B: CALCULATION OF SIMPLE INTEREST

Interest is computed on a *simple basis* if it is paid only on the principal of the loan. *Compound* interest is paid on accumulated loan interest as well as on the principal. Thus, if a sum of money (X_0) were borrowed at an annual interest rate (i) and repaid at the end of (n) years with accumulated interest, the total sum repaid (FV_n or *Future Value* at the end of Year n) is determined as follows:

(2.1) $$\mathbf{FV_n = X_0(1 + n \times i)}$$

The subscripts (n) and (0) merely designate time; they do not imply any arithmetic function. The product ($n \times i$) when multiplied by X_0 reflects the value of interest payments to be made on the loan; the value (1) accounts for the fact that the principal of the loan must be repaid. If the loan duration includes some fraction of a year, the value of (n) will be fractional; e.g., if the loan duration were one year and three months, (n) would be 1.25. The total amount paid (or, the Future Value of the loan) will be an increasing function of the length of time the loan is outstanding (n) and the interest rate (i) charged on the loan. For example, if a consumer borrowed $1000 at an interest rate of 10% for one year, his total repayment would be $1100, determined from Equation 2.1 as follows:

$$FV_1 = \$1000(1 + 1 \times .1) = \$1000 \times 1.1 \times \$1100$$

If the loan were to be repaid in two years, its future value would be determined as follows:

$$FV_2 = \$1000(1 + 2 \times .1) = \$1000 \times 1.2 = \$1200$$

Continuing our example, if the loan were to be repaid in five years, its future value would be:

$$FV_5 = \$1000(1 + 5 \times .1) = \$1000 \times 1.5 = \$1500$$

The longer the duration of a loan, the higher will be its future value. Thus, the longer lenders must wait to have their money repaid, the greater will be the total interest payments made by borrowers.

2.C: CALCULATION OF COMPOUND INTEREST

Interest is computed on a compound basis when a borrower must pay interest on not only the loan principal, but on accumulated interest as well. If interest must accumulate for a full year before it is compounded, the Future Value of such a loan is determined with Equation (2.2):[1]

(2.2) $$\mathbf{FV_n = X_0(1 + i)^n}$$

For example, if an individual were to deposit \$1000 into a savings account paying annually compounded interest at a rate of 10% (here, the bank is borrowing money), the future value of the account after five years would be \$1610.51, determined by Equation 2.2 as follows:

$$FV_5 = \$1000(1+.1)^5 = \$1000 \times 1.1^5 = \$1000 \times 1.61051 = \$1610.51$$

Notice that this sum is greater than the future value of the loan (\$1500) when interest is not compounded.

The compound interest formula can be derived intuitively from the simple interest formula. If interest must accumulate for a full year before it is compounded, then the future value of the loan after one year is \$1100, exactly the same sum as if interest had been computed on a simple basis:

(2.3) $$FV_n = X_0(1+ni) = X_0(1+1 \times i) = X_0(1+i)^1 = \$1000(1+.1) = \$1100$$

The future values of loans where interest is compounded annually and when interest is computed on an annual basis will be identical only when (n) equals one. Since the value of this loan is \$1100

[1]Readers who are unfamiliar with exponent and subscript notation may wish to read the Elementary Mathematics Review available on the course web site.

after one year and interest is to be compounded, interest and future value for the second year will be computed on the new balance of \$1100:

(2.4) $$FV_2 = X_0(1+1\times i)(1+1\times i) = X_0(1+i)(1+i) = X_0(1+i)^2 ,$$

$$FV_2 = \$1000\ (1 + .1)(1 + .1) = \$1000\ (1 + .1)^2 = \$1210$$

This process can be continued for five years:

$$FV_5 = \$1000(1+.1)(1+.1)(1+.1)(1+.1)(1+.1) = \$1000(1+.1)^5 = \$1610.51$$

More generally, the process can be applied for a loan of any maturity. Therefore:

(2.5) $$FV_n = X_0(1+i)(1+i)^{\cdot\cdot\cdot}(1+i) = X_0(1+i)^n ,$$

$$FV_n = \$1000\ (1 + .1)(1 + .1)^{\cdot\cdot\cdot}(1 + .1) = \$1000\ (1 +.1)^n$$

APPLICATION 2.1: The Purchase of Manhattan Island

In 1626, Dutchman Peter Minuit purchased the island of Manhattan from the Wappinger Indians for approximately \$24 in trinkets and other merchandise. This island was to become the center of New York City, and the location of some of the most valuable real estate in the world. Suppose the Indians had sold their merchandise for \$24, and invested it at an annual rate of 6% compounded once per year. What would be the value of their investment in 2014?

Assuming the original investment amount (X_0 = \$24) is invested for n = 388 years at an annual rate of .06, we obtain the following:

$$FV_n = \$24(1+.06)^{388} = \$158{,}083{,}653{,}510$$

Thus, the over 158 billion dollars that this investment would be worth today would be worth several times the total value of the land comprising the island.

2.D. FRACTIONAL PERIOD COMPOUNDING OF INTEREST

In the previous examples, interest is compounded annually; that is, interest must accumulate at the *stated rate i* for an entire year before it can be compounded or re-compounded. In many savings accounts and other investments, interest can be compounded semiannually, quarterly or even daily. If interest is to be compounded more than once per year (or once every fractional part of a year), the future value of such an investment will be determined as follows:

(2.6) $$\mathbf{FV_n = X_0\ (1 + i/m)^{mn}} ,$$

where interest is compounded (m) times per year. The interpretation of this formula is fairly straightforward. For example, if (m) is 2, then interest is compounded on a semiannual basis. The semiannual interest rate is simply (i/m) or (i/2). If the investment is held for (n) periods, then it is held for (2n) semiannual periods. Thus, we compute a semiannual interest rate (i/2) and the number of semiannual periods the investment is held (2n). If $1000 were deposited into a savings account paying interest at an annual rate of 10% compounded semiannually, its future value after five years would be $1628.89, determined as follows:

$$FV_5 = \$1000(1 + .1/2)^{2\times5} = \$1000(1.05)^{10} = \$1000(1.62889) = \$1628.89$$

Notice that the semiannual interest rate is 5% and that the account is outstanding for ten six-month periods. This sum ($1628.89) exceeds the future value of the account if interest is compounded only once annually ($1610.51). In fact, the more times per year interest is compounded, the higher will be the future value of the account. For example, if the interest on the same account were compounded monthly (12 times per year), the account's future value would be $1645.31:

$$FV_5 = \$1000(1 + .1/12)^{12\times5} = \$1000(1.008333)^{60} = \$1645.31$$

The monthly interest rate is .008333 and the account is open for (mn) or 60 months. With daily compounding, the account's value would be $1648.60:

$$FV_5 = \$1000(1 + .1/365)^{365\times5} = \$1648.60$$

Therefore, as (m) increases, future value increases. However, this rate of increase in future value becomes smaller with larger values for (m); that is, the increases in (FV_n) induced by increases in (m) eventually become quite small. Thus, the difference in the future values of two accounts where interest is compounded hourly in one and every minute in the other may actually be rather trivial. Figure 2.2 depicts the impact of compounding frequency on future values. Notice in Figure 2.2 that increasing the number of periods for compounding increases the future value of loan amounts; however, this rate of increase occurs at a decreasing rate.

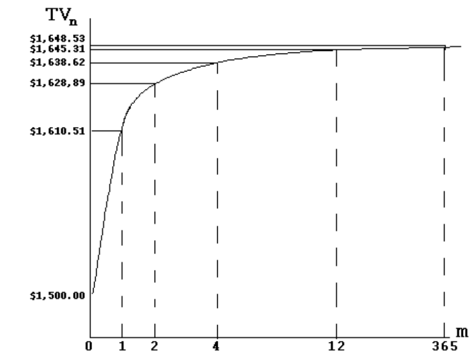

Figure 2.2: Effects of Compounding on Terminal Value

2.E. CONTINUOUS COMPOUNDING OF INTEREST

If interest were to be compounded an infinite number of times per period, we would say that interest is compounded continuously. However, we cannot obtain a numerical solution for future value by merely "plugging" in ∞ for m in Equation 2.6 - calculators have no ∞ key. In the previous section, we saw that increases in (m) cause the future value of an investment to increase. As (m) approaches infinity, (FV_n) continues to increase, however at decreasing rates. More precisely, as (m) approaches infinity ($m \rightarrow \infty$), the future value of an investment can be defined as follows:

(2.7) $$FV_n = X_0 e^{in}$$

where (e) is the natural log whose value can be approximated at 2.718 or derived from the following:[2]

(2.8) $$e = \lim_{m \to \infty} (1 + \frac{1}{m})^m$$

That is, as (m) approaches infinity, the value of the limit in expression (2.8) approaches the number (e). Notice the similarity between Equations (2.6), (2.8) and (2.9). In fact, Equation (2.7) can be derived easily from Equations (2.6) and (2.9) which defines e^i as follows:

(2.9) $$e^i = \lim_{m \to \infty} (1 + \frac{i}{m})^m$$

[2]The natural log is reviewed in the Elementary Mathematics Review on the course web site.

In many calculations involving continuous compounding of interest, the value 2.718 serves as an approximation for the number (*e*).

If an investor were to deposit $1000 into an account paying interest at a rate of 10%, continuously compounded (or compounded an infinite number of times per year), the account's future value would be approximately $1648.72:

$$FV_5 = \$1000 \times e^{.1\times 5} = \$1000 \times 2.718^{.5} = \$1648.72$$

The future value of this account exceeds only slightly the value of the account if interest were compounded daily. Also note that continuous compounding simply means that interest is compounded an infinite number of times per time period.

Years to maturity (n)	Future Value Simple Interest	Future Value Compounded Annually	Future Value Compounded Monthly	Future Value Compounded Daily	Future Value Compounded Continuously
1	110	110	110.47	110.52	110.52
2	120	121	122.04	122.14	122.14
3	130	133.31	134.81	134.98	134.99
4	140	146.41	148.94	149.17	149.18
5	150	161.05	164.53	164.86	164.87
10	200	259.37	270.70	271.79	271.83
20	300	672.75	732.81	738.70	738.91
30	400	1,744.94	1,983.74	2,007.73	2,008.57
50	600	11,739.09	14,536.99	14,831.16	14,841.40

TABLE 2.1: Future Values of accounts with initial $100 deposits at 10% interest

2.F. FUTURE VALUES OF ANNUITIES

An *annuity* is defined as a series of identical payments made at equal intervals. If payments are to be made into an interest bearing account, the future value of the account will be a function of interest accumulating on deposits as well as the deposits themselves. For example, many individuals open Individual Retirement Accounts (I.R.A.'s) from which they may withdraw when they reach the age of fifty-nine and one half years. Consider an individual who makes a $2000 contribution to his I.R.A. at the end of each year for twenty years. All of his contributions receive a ten percent annual rate of interest, compounded annually. What will be the total value of this account, including accumulated interest at the end of the twenty-year period? The following equation can be used to evaluate the future value of this annuity:

(2.11)
$$FVA_n = \frac{X}{i}\left((1+i)^n - 1\right)$$

where (X) is the annual contribution made at the end of each year by the investor to his account, (i) is the interest rate on the account and FVA is the future value of the annuity. This future value annuity equation can be used whenever identical periodic contributions are made toward an account. This future value equation 2.11 is derived in Equation Box 2.1. Such derivations are important because they are so frequently necessary for obtaining models for valuations of repetitive cash flows. In any case, we determine the future value of this individual's I.R.A. to be $114,550 as follows:

$$FVA_n = \frac{\$2{,}000}{.1}\left((1+.1)^{20} - 1\right) = \$114{,}550$$

Application Box 2.3 provides an interesting example involving the analysis of an annuity paid into an Individual Retirement Account.

Note that each of the calculations in the I.R.A. example assumes that cash flows are paid at the end of each period. If, instead, cash flows were realized at the beginning of each period, the annuity would be referred to as an *annuity due*. The annuity due would generate an extra year of interest on each cash flow. Hence, the future value of an annuity due is determined by simply multiplying the future value annuity formula by (1+i):

(2.12)
$$FVA_{n,Due} = \frac{X}{i}\left((1+i)^n - 1\right)(1+i) = \frac{X}{i}\left((1+i)^{n+1} - (1+i)\right)$$

From the example above, we find that the future value of the individual's I.R.A. is $126,005 if payments to the I.R.A. are made at the beginning of each year:

$$FVA_{n,Due} = \frac{\$2{,}000}{.1}\left((1+.1)^{21} - (1+.1)\right) = \$126{,}005$$

In 1981, shortly after the Individual Retirement Account (I.R.A.) was signed into effect by President James Carter, banks all over the United States were advertising "millionaire" accounts. One bank pitch would proceed similar to the following:

> "If you were to deposit $2,000 at the end of each year until the age of sixty five, you will have accumulated over $1,000,000 towards your retirement."

This pitch would be accompanied by a picture a relaxed retiree on his yacht sipping champaign and reading the *Wall Street Journal*. The ad would also mention that one should start saving at the age of 28 (that is, accumulate savings for 38 years), and continue to draw interest at the then prevailing rate of 12%. As we see from the following equation, the dollar amounts claimed by the bank making this advertisement were quite true:

$$FV_{38} = \$2{,}000/.12 \times [(1+.12)^{38} - 1] = \$1{,}087{,}197$$

However, the bank neglected to mention what would happen if interest rates dropped below 12% on such accounts and the impact that inflation would have on $1,000,000 over a thirty eight year period.

Application Box 2.3: I.R.A.'s and Millionaires

The future value annuity factor (fvaf) described in Section 2.F is used to determine the future value of an annuity. This annuity is a series of equal payments made at identical intervals. The future value annuity factor may be derived through the use of a simple algebraic technique known as a geometric expansion. This technique is very useful when a large number of repetitive computations must be performed, as is often the case in finance. The geometric expansion enables us to reduce a repetitive expression requiring many calculations to an expression which can be computed much more quickly. Consider the case where we wish to determine the future value of an account based on a payment of X made at the end of each year t for n years where the account pays an annual interest rate equal to i:

(1) $$FVA = X[(1+i)^{n-1} + (1+i)^{n-2} +...+ (1+i)^2 + (1+i)^1 + 1]$$

Thus, the payment made at the end of the first year accumulates interest for a total of (n-1) years, the payment at the end of the second year accumulates interest for (n-2) years and so on. Clearly, determining the future value of this account will be very time consuming if n is large. The first step in the geometric expansion is to multiply both sides of Equation (1) by (1+i):

(2) $$FVA(1+i) = X[(1+i)^n + (1+i)^{n-1} +...+ (1+i)^3 + (1+i)^2 + (1+i)]$$

The second step in the geometric expansion is to subtract Equation (1) from Equation (2) to obtain:

(3) $$FVA(1+i) - FVA = X[(1+i)^n - 1]$$

Notice that the subtraction led to the cancellation of many terms, reducing the equation we wish to compute to a much more manageable size. Finally, we rearrange terms in Equation (3) to obtain Equations (4) and (2.11):

(4) $$FVA\times 1 + FVA\times i - FVA = X[(1+i)^n - 1] = FVA\times i = X[(1+i)^n - 1]$$

(2.11) $$FVA = \frac{X[(1+i)^n - 1]}{i}$$

Practicing derivations such as this is an excellent way to understand the intuition behind financial formulas. Understanding the derivations is necessary in order to be able to modify the formulas for a variety of more complex (and realistic) scenarios. Appendix A.2.b at the end of this chapter provides a more general discussion on the geometric expansion procedure.

Derivation Box 2.1: Deriving Annuity Future Values

2.G: CONCLUSION

In this chapter, methods were presented for calculating future values of accounts when interest is computed on a simple basis, compounded annually, more than once annually and continuously. Perhaps the most useful of the equations presented here is (2.6):

$$FV_n = X_0(1 + i/m)^{mn}$$

Although this formula can be used when interest is compounded more than once per year, the formula can also be used to determine the future value of an account when interest must accumulate for a full year before it is compounded. In this case, we need only to set (m) equal to one. Equation (2.6) can also be used to estimate future value when interest is continuously compounded. This is done by allowing (m) to equal some very large number (perhaps 100,000).

Table 2.1 provides insight on the impact of compounding over time. Notice first that as the number of years to account maturity (n) increases, future value (FV_n) increases. When interest is compounded, the rate of increase in FV_n increases as n increases. Second, note also that as the number of compounding intervals (m) increases, future value increases. However, the rate of increase in FV_n decreases as m increases. This relationship is apparent from Figure 2.2. Finally, note that more years to account maturity increases the impact that the number of compounding intervals has on FV_n. Nonetheless, the equivalent annual rate (EAR) remains useful for comparisons of investments when their numbers of compounding intervals differ.

This chapter also offered an annuity expression for computing future values of annuities. This expression requires fewer computations to obtain future value when the number of deposits (n) is large. The derivation box for the terminal value annuity expression will provide insight into how more complicated finance formulas are obtained. Understanding such derivations is most useful for practitioners who encounter many situations where appropriate formulas are not available and must be "invented."

In this chapter, computations were concerned primarily with future value. Other interest-related topics such as amortization and bond yields will be discussed in Chapters 3 and 4.

QUESTIONS AND PROBLEMS

2.1. Why do interest rates charged by banks for the purchase of automobiles tend to exceed interest rates paid on savings accounts? Why are home loan (mortgage) interest rates usually lower than interest rates charged credit card customers?

2.2. The Williams Company has borrowed $10,500 at an annual interest rate of nine percent. How much will be a single lump sum repayment in eight years including both principal and interest accumulated on a simple basis? That is, what is the future value of this loan?

2.3. The Cobb Company has issued ten million dollars in ten percent coupon bonds maturing in five years. Interest payments on these bonds will be made semi-annually.

a. How much are Cobb's semi-annual interest payments?
b. What will be the total payment made by Cobb on the bonds in each of the first four years?
c. What will be the total payment made by Cobb on the bonds in the fifth year?

2.4. What would be the lump sum loan repayment made by the Williams Company in Problem 2.2 if interest were compounded:

a. annually?
b. Semiannually?
c. monthly?
d. daily?
e. continuously?

2.5. The Speaker Company has the opportunity to purchase a five-year $1000 certificate of deposit (C.D.) paying interest at an annual rate of 12%, compounded annually. The company will not withdraw early any of the money in its C.D. account. Will this account have a greater future value than a five-year $1000 C.D. paying an annual interest rate of 10%, compounded daily?

2.6. The Waner Company needs to set aside a sum of money today for the purpose of purchasing for $10,000 a new machine in three years. Money used to finance this purchase will be placed in a savings account paying interest at a rate of eight percent. How much money must be placed in this account now to assure the Waner company $10,000 in three years if interest is compounded yearly?

2.7. A given savings account pays interest at an annual rate of 9% compounded quarterly. Find the annual percentage yield (APY) for this account.

2.8. Assuming no withdrawals or additional deposits, how much time is required for $1000 to double if placed in a savings account paying an annual interest rate of 10% if interest were:

a. computed on a simple basis?

b. compounded annually?
c. compounded monthly?
d. compounded continuously?

2.9.* Assume that you are advising a twenty-three year-old client with respect to personal financial planning. Your client wishes to save, become a millionaire and then retire. Your client intends to open and contribute to a tax deferred Individual Retirement Account each year until he retires with $1,000,000 in that account.

a. If your client were to deposit $2000 at the end of each year into his I.R.A., for how many years must he wait until he retires with his $1,000,000? Assume that the account will pay interest at an annual rate of 10%, compounded annually.
b. What would your answer to part a be if the interest rate were 12%?
c. What would the client's annual payment have to be if he wished to retire at the age of forty with $1,000,000? Assume that the client will make deposits at the end of each year for 17 years at an annual interest rate of 10% and that his I.R.A. will be supplemented with another type of retirement account known as a 401(k) so that his total annual tax deferred deposits can exceed $2,000.
d. What would your answer to part c be if your client were willing to wait until he is fifty to retire?
e. What would your answer to part d be if your client were able to make deposits into an account paying interest at an annual rate of 12%?
f. What would your answers to parts a, c and d be in the annual interest rate were only 4%?
g. If the annual inflation rate for the next fifty years were expected to be 3%, what would be the purchase power of $1,000,000 in 17 years? In 27 years?
h. What would be your answers to part g be if the inflation rate were expected to equal 9%?

CHAPTER 2 APPENDIX

2.A: GEOMETRIC EXPANSIONS

Here, we introduce the concept of the geometric expansion as a technique to simplify a polynomial consisting of a repetitive series of terms. These terms, arranged in a series of terms with a single variable and exponents arranged in descending order of exponents is called a geometric series. A geometric expansion is an algebraic procedure used to simplify a geometric series. This procedure is most useful when the number of terms is large. Suppose one intended to solve the following finite geometric series for S:

(A) $$S = c + cx + cx^2 + cx^3 + \ldots + cx^n$$

In this series, c is a constant or parameter and x is a quotient or variable. If n is large, direct calculations on this series may be time consuming and repetitive. Simplifying the series to reduce the number of terms may save a significant amount of time performing routine calculations. The geometric expansion is a two-stage procedure:

1. First, multiply both sides of the equation by the quotient:

(B) $$Sx = cx + cx^2 + cx^3 + cx^4 + \ldots + cx^{n+1}$$

This first step is intended to obtain a very similar type of expression with repetitive terms that will be eliminated in the second step.

2. Second, to eliminate these repetitive terms, subtract the above product (B) from the original equation (A) and then simplify the result:

(C) $$\begin{aligned} Sx - S = {} & cx + cx^2 + cx^3 + cx^4 + \ldots + cx^{n+1} \\ & - c - cx - cx^2 - cx^3 - \ldots - cx^n \end{aligned}$$

The following simplification completes the geometric expansion. Notice the set of terms that should cancel when we simplify:

(D) $$Sx - S = -c + cx^{n+1}$$

(E) $$S(x - 1) = c(x^{n+1} - 1)$$

Continue the process of simplification by dividing both sides by (x - 1):

(F) $$S = c\left(\frac{x^{n+1} - 1}{x - 1}\right) \text{for } x \neq 1.$$

Consider the following example where we set x to equal (1+i). Equations G and H will be identical:

(G) $$S = c + c(1+i) + c(1+i)^2 + c(1+i)^3 + \ldots + c(1+i)^n$$

(H) $$S = c\left(\frac{1 - (1+i)^{n+1}}{1-(1+i)}\right) = c\frac{(1+i)^{n+1}-1}{i}$$

Thus, any geometric series where $x \neq 1$ can be simplified with the following right-hand side formula:

(I) $$S = c + cx + cx^2 + cx^3 + \ldots + cx^{n-1} = c\frac{x^n-1}{x-1}$$

Geometric expansions are most helpful in time value mathematics with many periods and in situations involving series of potential outcomes with associated probabilities. Such situations occur very frequently in finance. The geometric expansion procedure can save substantial amounts of computation time for problems involving these situations.