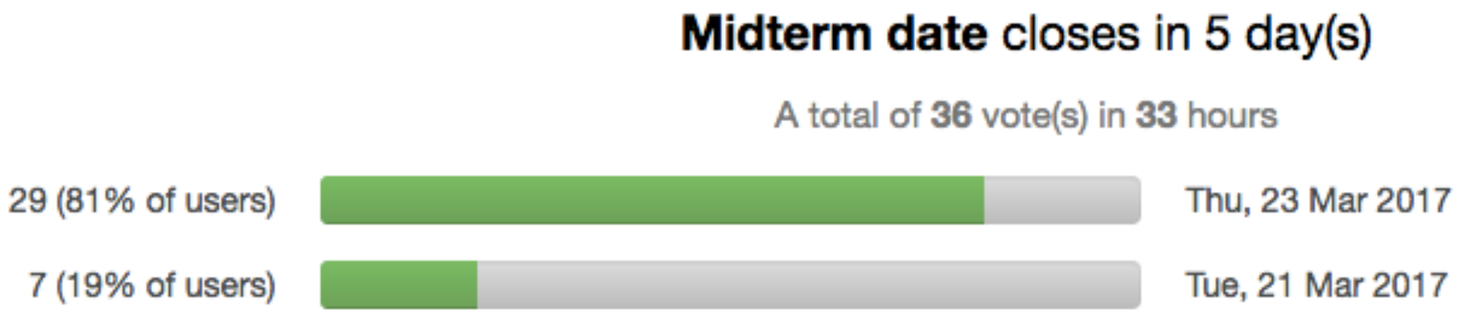
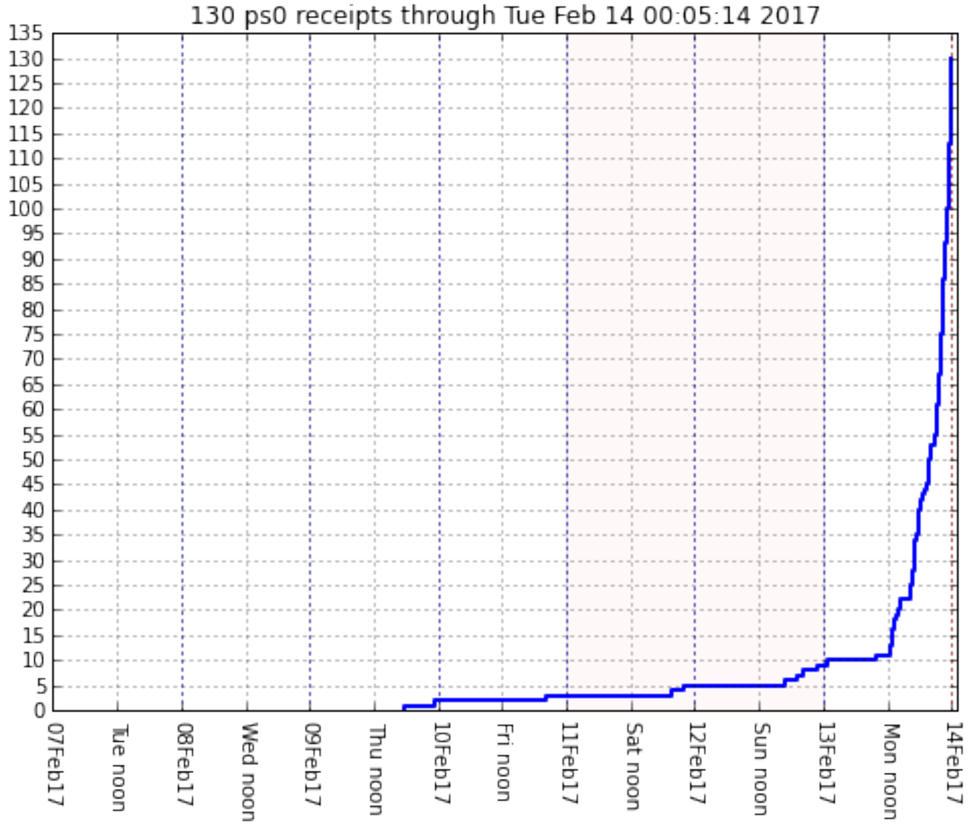
Info 2950, Lecture 6

14 Feb 2017

More Programming and Statistics Boot Camps?

Prob Set 2: due Fri night 24 Feb





Brief Summary

- Expectation value: $E[X] = \sum_{s \in S} p(s) X(s)$

- Variance: $V[X] = \sum_{s \in S} p(s)(X(s) - E[X])^2$
$$= E[X^2] - (E[X])^2$$

- Standard deviation: $\sigma[X] = \sqrt{V[X]}$

- For X a sum of random variable $X = \sum_i X_i$, the expectation always satisfies:
$$E[X] = \sum_i E[X_i]$$

- If (and only if) the variables X and Y are *independent*, then
$$E[XY] = E[X]E[Y]$$

- If (and only if) all the variables X_i are *independent*, then
$$V[X] = \sum_i V[X_i]$$

Bernoulli Trial

A Bernoulli trial is a trial with two possible outcomes:

"success" with probability p,

and "failure" with probability $1 - p$.

$$\text{Probability } (r \text{ successes in } N \text{ trials}) = \binom{N}{r} p^r (1-p)^{N-r}$$

To analyze a succession of N Bernoulli trials,

let random variable $X_i = 0$ or 1 represent the result of the ith trial,

where $X_i = 1$ if the trial is a "success" (with probability p). Then

$$E[X_i] = (1-p) \cdot 0 + p \cdot 1 = p$$

Now let $X = X_1 + X_2 + \ldots + X_N$ count the total number of successes.

Using linearity of expected value, the mean satisfies

$$\boxed{E[X] = \sum_{i=1}^{N} E[X_i] = Np}$$

Now recall that in general $V[X_i] = E[X_i^2] - E[X_i]^2$, and here

$$E[X_i^2] = (1-p)\cdot 0^2 + p \cdot 1^2 = p$$

so using as well $E[X_i]^2 = p^2$ we find

$$V[X_i] = E[X_i^2] - E[X_i]^2 = p - p^2 = p(1-p)$$

Finally, since the X_i are independent random variables, it follows that the variance satisfies

$$\boxed{V[X] = \sum_{i=1}^{N} V[X_i] = Np(1-p)}$$

and hence the standard deviation is $\sigma = \sqrt{V[X]} = \sqrt{Np(1-p)}$.

For $p = 1/2$ and $N = 3$, these give

$$E[X] = Np = 3/2 \qquad\qquad V[X] = Np(1-p) = 3/4$$

reproducing the result of the coin flip example from earlier:

Example: again flip a coin 3 times, and let X be the number of tails.

$$E[X^2] = \frac{1}{8}0^2 + \frac{3}{8}1^2 + \frac{3}{8}2^2 + \frac{1}{8}3^2 = 3$$

so $V[X] = E[X^2] - E[X]^2 = 3 - (3/2)^2 = 3/4$.

Here $X = X_1 + X_2 + X_3$, where X_i is the number of tails (0 or 1) for the ith roll,
then the X_i are independent variables with $E[X_i] = 1/2$
and $E[X]$ is additive: $E[X] = \sum_i E[X_i] = 3 \cdot \frac{1}{2}$

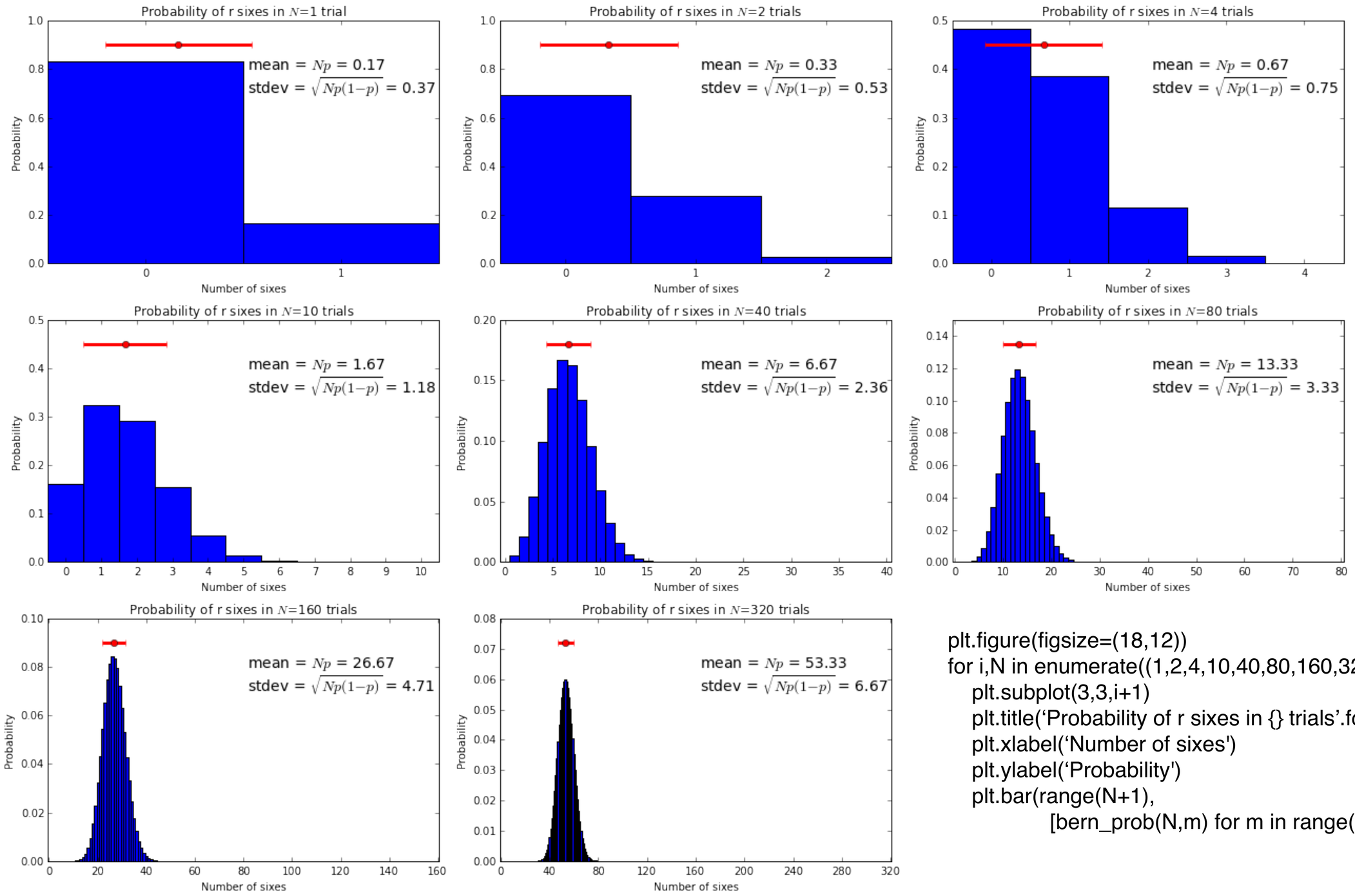
Again the results

$$E[X] = Np \qquad \text{and} \qquad \sigma[X] = \sqrt{V[X]} = \sqrt{Np(1-p)}$$

explain the observation that the probability gets more sharply peaked as the number of trials increases, since the width of the distribution (σ) divided by the average $E[X]$ behaves as

$$\frac{\sigma[X]}{E[X]} = \frac{\sqrt{Np(1-p)}}{Np} = \frac{\sqrt{N}}{N}\sqrt{\frac{1-p}{p}} \sim \frac{1}{\sqrt{N}}$$

a decreasing function of N (for any fixed p).

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plt.figure(figsize=(18,12))
for i,N in enumerate((1,2,4,10,40,80,160,320)):
    plt.subplot(3,3,i+1)
    plt.title('Probability of r sixes in {} trials'.format(N))
    plt.xlabel('Number of sixes')
    plt.ylabel('Probability')
    plt.bar(range(N+1),
            [bern_prob(N,m) for m in range(N+1)])
```

By the "central limit theorem" (won't be proven here), many such distributions always tend for sufficiently large number of trials to a universal form

$$P(x) \approx \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

known as the "gaussian" or "normal" distribution.

Here $x = r =$ number of successes and mean $\mu = E[X]$.

[This is properly normalized, with $\int_{-\infty}^{\infty} \mathrm{d}x\, P(x) = 1$, and also has

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x P(x) = \mu \qquad \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 P(x) = \sigma^2 + \mu^2$$

so the above distribution has mean μ and variance σ^2.]

Setting $\mu = Np$ and $\sigma = \sqrt{Np(1-p)}$ for $p = 1/6$ thus gives a good approximation to the distribution of successful rolls of 6 for large number of trials in the example above.

Central limit theorem:

Consider a set of N independent random variables X_i,
with arbitrary probability distributions $p_i(x)$,
with means μ_i and variances σ_i^2,

Consider the sum $X = \sum_{i=1}^{N} X_i$, then for sufficiently large N,
the probability distribution of X will tend to a gaussian (normal) distribution with mean:

$$\mu = \sum_{i=1}^{N} \mu_i$$

and variance:

$$\sigma^2 = \sum_{i=1}^{N} \sigma_i^2$$