



Consider the Markov chain depicted above. The only recurrent states are 1 and 5, each trivially recurrent since it can only go to itself, and hence each forms its own recurrent class. (The stationary distributions have  $p_1 = 1$  and  $p_5 = 1$ .)

Assume the Markov chain is in state 0 just before the first step, some questions:

(a) What is the probability of being in state 7 after  $n$  steps?

The unique path is a first step from 0 to 7, then  $n - 1$  steps from 7 to itself, hence the probability is  $.5(.4)^{n-1}$ .

(b) What is the probability of reaching state 2 for the first time on the  $k^{\text{th}}$  step?

The unique path for this is a first step from 0 to 7, then  $k - 2$  steps from 7 to itself, then a step from 7 to 2, hence the probability is  $(.5)(.4)^{k-2}(.2)$

(c) What is the probability of never reaching state 1 ?

The only way to avoid state 1 is to follow the path from  $0 \rightarrow 7 \rightarrow 4$ . From state 7 it's twice as likely to go to state 4 as to state 2, therefore  $2/3$  of the time one will eventually end up in the direction of state 4 from state 7. Since the probability of going from state 0 to state 7 is  $1/2$ , the overall probability of going from state 0 to state 4, and never reaching state 1, is  $(1/2) \cdot (2/3) = 1/3$ .

[Another way to reproduce the above is to note that the overall probability to reach state 2 from state 7 is to sum over paths that return  $k$  times to state 7 (and use  $\sum_{n=0}^{\infty} r^n = 1/(1 - r)$ ):  $p_{72} \sum_{k=0}^{\infty} p_{77}^k = \frac{p_{72}}{1-p_{77}} = .2/.6 = 1/3$ . Similarly, the probability to eventually reach state 4 from state 7 is  $p_{74} \sum_{k=0}^{\infty} p_{77}^k = \frac{p_{74}}{1-p_{77}} = .4/.6 = 2/3$ .]

(d) What is the expected number of steps until hitting state 5?

If the probability of leaving a state is  $p$ , then the probability of leaving after exactly  $n$  steps is  $(1 - p)^{n-1}p$ . With  $q = 1 - p$ , the expectation value for the number of steps is

$$E[n] = \sum_{n=0}^{\infty} n q^{n-1} p = p \frac{\partial}{\partial q} \sum_{n=0}^{\infty} q^n = p \frac{\partial}{\partial q} \frac{1}{1 - q} = p \frac{1}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

Thus the expected numbers of steps to remain at states 7,4,6 respectively are  $1/.6 = 5/3$ ,  $1/.8 = 5/4$ ,  $1/.3 = 10/3$ , and the expected number of steps until hitting state 5 is thus  $1 + 5/3 + 5/4 + 10/3 = 7 1/4$ . [Note that the number of steps to leave a state is also called the "waiting time". It gives the expected number of times to flip a first H as  $1/p = 1/(1/2) = 2$ , and the expected number of times to roll a first 1 as  $1/p = 1/(1/6) = 6$ .]

(e) What is the probability of eventually hitting state 5?

This is the same probability of  $1/3$  as (c), since any path that doesn't reach 1 eventually hits 5. More specifically, the probability of going from state 0 to state 7 is  $1/2$ , the probability of eventually going from  $7 \rightarrow 4$  is  $2/3$ , and from state 4 one eventually hits state 5, so the overall probability of going from state 0 to state 5 is  $p(0 \rightarrow 5) = (1/2) \cdot (2/3) = 1/3$ .

(f) What is the probability of being in state 4 after two steps, given that one is in state 5 after 8 steps?

This is given by the joint probability of being in state 4 after two steps *and* being in state 5 after 8 steps, divided by the overall probability of being in state 5 after 8 steps

$$p(\text{state 4 after 2 steps} \mid \text{state 5 after 8 steps}) = \frac{p(\text{state 4 after 2 steps}, \text{state 5 after 8 steps})}{p(\text{state 5 after 8 steps})}.$$

The probability of being in state 5 after 8 steps is

$$p(\text{state 5 after 8 steps}) = p_{07} p_{74} p_{46} p_{65} \sum_{\substack{\{i,j,k,\ell\} \\ i+j+k+\ell=4}} p_{77}^i p_{44}^j p_{66}^k p_{55}^\ell = .5 \cdot .4 \cdot .8 \cdot .3 \cdot (6.6759) \approx .32,$$

where the sum is over all ways of returning to states 7,4,6, and 5 for an overall total of four times. (Note in this case that  $p_{55} = 1$  so it doesn't matter when state 5 is reached or how many times one returns to it.) There are many ways to calculate the above sum. One way is to consider the product

$$(1 + .6x + .6^2x^2 + .6^3x^3 + .6^4x^4)(1 + .2x + .2^2x^2 + .2^3x^3 + .2^4x^4)(1 + .7x + .7^2x^2 + .7^3x^3 + .7^4x^4) \\ = 1 + 1.5x + 1.57x^2 + 1.419x^3 + 1.1869x^4 + \dots$$

The powers of  $x$  count the sum of the number of steps back to states 7,4,6, so returning to those states four or fewer times corresponds to  $1 + 1.5 + 1.57 + 1.419 + 1.1869 = 6.6759$ .

The joint probability of being in state 4 after two steps and being in state 5 after 8 steps is given by the same sum as above except with  $i = 0$ , since there can be no transitions from state 7 back to itself if state 4 is reached in just 2 steps. This gives

$$p(\text{state 4 after 2 steps}, \text{state 5 after 8 steps}) = \\ = p_{07} p_{74} p_{46} p_{65} \sum_{\{j,k,\ell \mid j+k+\ell=4\}} p_{44}^j p_{66}^k p_{55}^\ell = .5 \cdot .4 \cdot .8 \cdot .3 \cdot (3.3825) \approx .162.$$

In this case the sum was given by considering the product

$$(1 + .2x + .2^2x^2 + .2^3x^3 + .2^4x^4)(1 + .7x + .7^2x^2 + .7^3x^3 + .7^4x^4) \\ = 1 + .9x + .67x^2 + .477x^3 + .3355x^4 + \dots$$

where the power of  $x$  counts the combined number of steps back to 4 and 6, and for a total of four or fewer times corresponds to the sum  $1 + .9 + .67 + .477 + .3355 = 3.3825$ .

The result for the conditional probability is

$$p(\text{state 4 after 2 steps} \mid \text{state 5 after 8 steps}) = \frac{.5 \cdot .4 \cdot .8 \cdot .3 \cdot (3.3825)}{.5 \cdot .4 \cdot .8 \cdot .3 \cdot (6.6759)} \approx .506.$$