Sets

Definition. *A* **set** *is a collection of objects.*

The objects of a set are referred to as the *elements* of the set. If S is a set and x is some element in S we write $x \in S$. To show that some object x is not in a set S, we write $x \notin S$. The objects of a set can be anything and the order and multiplicity of the elements does not change the set.

Example: we could have a set $X = \{1, 2, 3, 4, 5\}$ or $C = \{$Ithaca, Boston, Chicago$\}$. Also, the objects of a set do not have to seem related. We could have a set Stuff $= \{1$, snow, Cornell, $y\}$. A set can also have no elements, this is called the empty set and written $\emptyset$.

We do not need to define a set by listing all the elements in the set. A set can be defined by a rule or equation.

Example: define E to be the set of even numbers. To express this we write $E = \{\, x \mid x \text{ is even}\}$ or $\{\, x \mid x + 1 \text{ is odd}\}$.

The *cardinality* $|S|$ of a set is the number of elements in the set.

Example: $|X| = 5$, $|C| = 3$, $|\text{Stuff}| = 4$ and $|\emptyset| = 0$. (Note that the set E is infinite. We we will not be considering the cardinalities of infinite sets in this class.)

A *subset* T of a set S is a set of elements all of which are contained in S. Note that T can be all of S. We write $T \subset S$ if we want to require that T is not equal to S, and say T is a *proper* subset. Otherwise we write $T \subseteq S$. The empty set is a subset of every set.

Example: $C' = \{$Ithaca, Chicago$\}$ is a proper subset of C and $X' = \{x \mid x$ is a whole number between 2 and 5$\}$ is a subset of X.

The *power set* $\mathcal{P}(S)$ of a set S is the set of all subsets of S.

Example: For the set $A = \{1, 2, 3\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Question: For a set S with n elements, what is $|\mathcal{P}(S)|$?

Set Operations

The *union* of two sets $A \cup B = \{\, x \mid x \in A \text{ or } x \in B\}$.

Example:

$X\cup$ Stuff $= \{1, 2, 3, 4, 5, \text{snow, Cornell}, y\}$.

$C \cup \emptyset = \{\text{Ithaca, Boston, Chicago }\}$.

$A \cup X = \{1, 2, 3, 4, 5\}$. In this case, $A \cup X = X$.

The *intersection* of two sets $A \cap B = \{\, x \mid x \in A \text{ and } x \in B\}$.

Example:

$X\cap$ Stuff $= \{1\}$.

$C \cap \emptyset = \emptyset$.

$X \cap E = \{2, 4\}$.

$A \cap X = \{1, 2, 3\}$. In this case, $A \cap X = A$.

The *difference* of two sets $A - B = \{\, x \mid x \in A \text{ and } x \notin B\}$.

Example:

$X-$ Stuff $= \{2, 3, 4, 5\}$.

Stuff $-X = \{\text{snow, Cornell}, y\}$.

$C - \emptyset = \{\text{Ithaca, Boston, Chicago}\}$.

$X - E = \{1, 3, 5\}$.

The *symmetric difference* $A \triangle B = \{\, x | \, x \in A \text{ or } x \in B, \text{ and } x \notin A \cap B\}$.

Example:

$X\triangle$ Stuff $= \{2, 3, 4, 5, \text{snow, Cornell}, y\}$.

$C \triangle \emptyset = \{\text{Ithaca, Boston, Chicago}\}$.

$X \triangle A = \{4, 5\}$.

The *Cartesian product* of two sets $A \times B = \{\, (x, y) \mid x \in A \text{ and } y \in B\}$.

Example:

$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

For two sets to be the same, they must have the same elements. Namely, $A = B$ says that for every element x, $x \in A$ if and only if $x \in B$. Equivalently, $A = B$ says that $A \subseteq B$ and $B \subseteq A$.