A tree is a connected graph with no cycles. A forest is a graph with each connected component a tree. A leaf in a tree is any vertex of degree 1.

Example Figure 11 shows a tree and a forest of 2 trees.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{tree_forest.png}
\caption{A tree and a forest.}
\end{figure}

Proposition. For any tree $T = (V, E)$ with $|V| = n$, $|E| = n - 1$.

Proof. Consider any leaf of $T$. This vertex is adjacent to exactly one edge. Remove this vertex and edge contributing 1 each to the number of vertices and edges. Continue removing leaf / edge pairs until we are left with just a single edge. A graph with a single edge has one more vertex than edge, hence the total number of edges is one less than the total number of vertices. \hfill \qed

A graph $G$ is planar if there exists an embedding of $G$ into the plane such that no two edges cross.

Example: The graph on 4 vertices with edges $(1, 2)$, $(2, 3)$, $(3, 4)$ and $(4, 1)$ is planar. Figure 12 shows this graph drawn with 1 edge crossing and with no edge crossings.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{planar_graph.png}
\caption{Two representations of the cycle of length 4.}
\end{figure}
The edges of a planar embedding of a graph divide the plane into regions. Let \( f \) be the number of regions of a planar graph, \( e \) the number of edges and \( v \) the number of vertices.

**Theorem.** *(Euler’s formula)* For any connected planar graph, \( v - e + f = 2 \).

**Proof.** We proceed by induction on the number of edges \( e \). Consider the case \( e = 1 \). There is only one such graph. This graph has \( v = 2, e = 1 \) and \( f = 1 \). Hence \( v - e + f = 2 \). Assume the formula holds for any connected planar graph on \( n \) edges. Consider a connected planar graph \( G \) with \( n + 1 \) edges, \( v \) vertices and \( f \) regions. Form \( G' \) with statistics \( e', v' \), and \( f' \) by removing any edge which results in another connected graph. In this case, \( v' = v \), \( e' = e - 1 \), and \( f' = f - 1 \). (why?) Therefore we have \( 2 = v' - e' + f' = v - (e - 1) + (f - 1) = v - e + f \).

In forming \( G' \), it could have been that removing any edge of \( G \) resulted in a disconnected graph. In this case, \( G \) is a tree. (why?) Using the proposition above, we know that for any tree \( v - e + f = v - (v - 1) + 1 = 2 \).

\[ \square \]

**Proposition.** For any connected planar graph with \( v \geq 3 \), \( e \leq 3v - 6 \).

**Proof.** Consider tracing out the boundary of any given region \( F \). Count the number of times we traverse an edge and call this the degree of \( F \). If we traced out every region of \( G \), we would traverse each edge exactly twice. Hence the sum of the degrees of all regions is exactly \( 2e \). Next note that each region has at least 3 edges on its boundary. Therefore we can conclude that \( 2e \geq 3f \). Using Euler’s formula we get: \( 2e \geq 3(2 - v + e) \) or \( e \leq 3v - 6 \).

\[ \square \]

We want to consider two common operations on a graph. The *deletion* of an edge in a graph is removing this edge from the graph. The *contraction* of an edge in a graph deletes the edge and identifies its endpoints to a common vertex. A *minor* of a graph \( G \) is any new graph formed from \( G \) by a series of deletion and contraction operations.

**Example** Figure 13 shows the deletion and contraction of the edge \((1, 2)\).
Theorem. A graph is planar iff it does not contain either graph of Figure 14 as a minor.

A proper coloring of a graph is a map $f$ from the vertices of a graph to $\{1, 2, 3, \ldots\}$ such that if $(v_i, v_j) \in E$ then $f(v_i) \neq f(v_j)$. The chromatic number of a graph $G$ is the minimum number of colors needed for a coloring of $G$.

Theorem. (4-color theorem) The chromatic number of any planar graph is less than or equal to 4.