Graphs

Definition. A graph $G$ is a pair $(V,E)$, where $V$ is a finite set and $E$ is a collection of unordered pairs in $V \times V$. The elements of $V$ are referred to as the vertices of the graph and the elements of $E$ are referred to as the edges.

A graph is an abstract structure, but immediately we associate a representation to any graph by drawing the vertices as points and the edges as lines.

Example Define a graph $G_1$ with $V = \{1, 2, 3, 4, 5\}$ and $E = (1,2) (1,3) (1,4) (2,4) (3,4) (1,5) (4,6)$. We represent $G$ with the picture in Figure 1.

![Figure 1: A graph with 5 vertices and 7 edges.](image)

We could have repeated elements in $E$. For example, we could include an edge twice. In this case, we say that $G$ has multiple edges. We could also have an edge between just one vertex. In this case, we call the edge a loop.

Example In Figure 2, $(3,3)$ is an edge and $(1,5)$ has been included twice. A simple graph is a graph with no loops and no multiple edges.

![Figure 2: A graph with loops and multiple edges](image)
In our definition of a graph, edges are unordered pairs. So the edge \((1, 4)\) is the same thing as the edge \((4, 1)\). This kind of graph is referred to as **undirected**. A **directed** graph is a graph whose edges are ordered pairs. In the representation of points and lines, we show this by using arrows. For an edge \((v_1, v_2)\), we draw an arrow beginning at \(v_1\) and ending at \(v_2\).

**Example** Figure 3 shows a directed graph with \(V = \{1, 2, 3, 4, 5, 6\}\) and \(E = (1, 2) (1, 3) (4, 1) (4, 2) (3, 4) (1, 5) (6, 4)\).

![Figure 3: A directed graph.](image)

A **subgraph** \(H = (V', E')\) of a graph \(G = (V, E)\) is a pair \(V' \subseteq V\) and \(E' \subseteq E\). We say that \(H\) is an **induced** subgraph of \(G\) if all the edges between the vertices in \(V'\) from \(E\) are in \(E'\).

**Example** Figure 4 shows two subgraphs of \(G_1\). The first subgraph is an induced subgraph. All edges between the vertices 2, 3, 4, and 6 that are in \(G_1\) are also in this graph. The second subgraph is not an induced subgraph because the edges \((2, 4)\) and \((1, 4)\) are missing.

![Figure 4: Subgraphs of \(G_1\).](image)
We say that two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are the same or isomorphic if there is a one to one and onto map \( f \) from \( V_1 \) to \( V_2 \) such that \((f(v_i), f(v_j))\) is an edge in \( G_2 \) if and only if \((v_i, v_j)\) is an edge in \( G_1 \).

**Example** The two graphs of Figure 5 are isomorphic. In this example, we can simply take the map \( f(1) = 1, f(2) = 2, \ldots, f(6) = 6 \). In general, if we have a graph whose vertices are labeled and we can find a map such that \( f(i) = i \) then we say that the graphs are isomorphic as labeled graphs.

![Figure 5: Two isomorphic graphs.](image)

We can change the labeling on one of the graphs so that they will no longer be isomorphic as labeled graphs (they are still isomorphic as unlabeled graphs).

**Example** In Figure 6 the labels 2 and 5 have been changed, but the graphs are still isomorphic.

![Figure 6: Two isomorphic graphs with different labellings.](image)
Two vertices $v_1$, $v_2$ are *adjacent or neighbors* if $(v_1, v_2) \in E$. Two edges are *adjacent* if they share a common vertex. A *path* from $v_1$ to $v_n$ in a graph is a sequence of adjacent edges such that $v_1$ is in the first edge of the sequence and $v_n$ is in the last edge of the sequence.

**Example** In Figure 7 the path $(5,1)(1,3)(3,4)(4,2)$ from 5 to 2 is highlighted.

![Figure 7: A path from vertex 5 to vertex 2.](image)

We say a graph is *connected* if for every pair of vertices there exists a path between them. All the graphs we have seen so far are connected. Figure 8 shows a graph that is not connected. A disconnected graph consists of multiple connected pieces called *components*.

![Figure 8: A disconnected graph with 2 connected components.](image)