Notes on derivation of power law for preferential attachment

First we define the preferential attachment process. Let $d_i(t)$ be the degree $\deg(v_i)$ of node v_i at time t. At time $t+1$ we add a new vertex v_{t+1} with m new edges to the earlier nodes v_i, with probability proportional to their degree $d_i(t)$:

$$Pr(\text{attaching to } v_i) = \frac{d_i(t)}{\sum_{j=1}^{n} d_j(t)} \ . \tag{1}$$

(You will implement this in the next programming assignment.)

To analyze this, we can approximate the probabilistic discrete time dynamics with a continuous time deterministic process, which turns out to capture the essential behavior of the degree distribution. We suppose that $d_i(t)$ depends on a continuous time t, with the boundary condition that node v_i is created at time t_i with degree m, so that

$$d_i(t_i) = m \ , \tag{2}$$

and as well the sum of the degrees $d_j(t)$ satisfies $\sum_{j=1}^{n} d_j(t) = 2mt$ (since by time t we've added mt edges, and as usual each contributes 1 to the degree of two nodes).

As new nodes and edges are added according to (1), the degree of v_i increases as

$$\frac{\partial}{\partial t} d_i(t) = \frac{m d_i(t)}{\sum_{j=1}^{n} d_j(t)} = \frac{d_i(t)}{2t} \tag{3}$$

(each has m independent chances). Rewritten as $\partial d_i(t)/d_i(t) = \partial t/2t$, this integrates to

$$\ln d_i(t) = \frac{1}{2} \ln t + \text{const} \ . \tag{4}$$

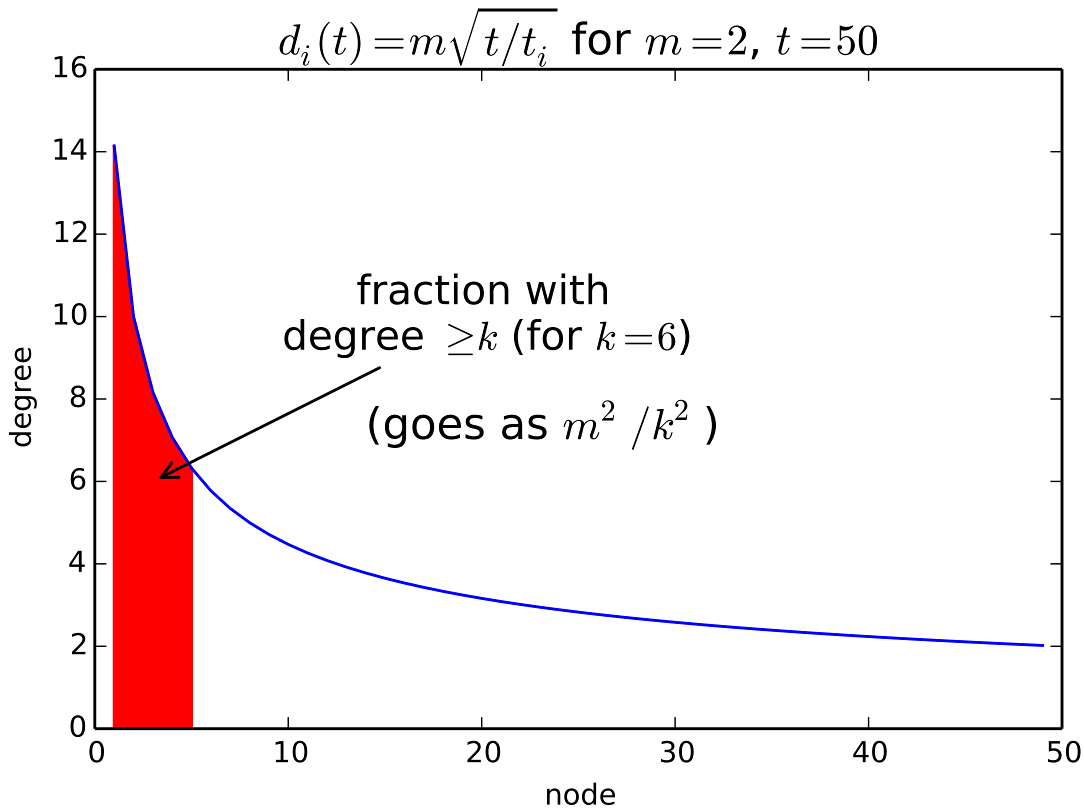
Exponentiating, we have $d_i(t) = Ct^{1/2}$, where the new constant C can be determined by the condition (2) and therefore (for $t \geq t_i$):

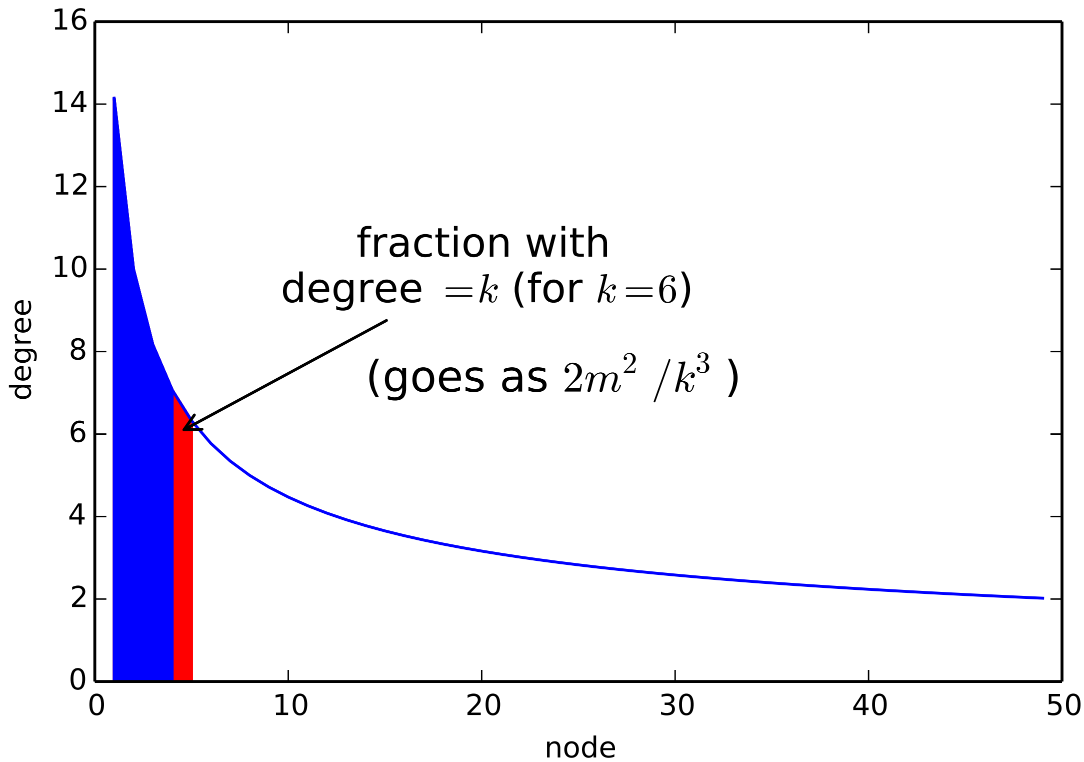
$$d_i(t) = m\sqrt{\frac{t}{t_i}} \ . \tag{5}$$

Now consider $F(k)$, the fraction of nodes that have degree $d_i(t)$ at least equal to k (where $k \geq m$ since nodes are created with degree m). Substituting (5) into the condition $d_i(t) \geq k$ gives $m\sqrt{t/t_i} \geq k$, or equivalently $t_i \leq (m^2/k^2)t$. So at any time t, the fraction of nodes with degree at least equal to k consists of those created in the first m^2/k^2 fraction of the time, and $F(k) = m^2/k^2$. In the discrete time version, the fraction of nodes with degree equal to k, $Pr(d_i(t) = k)$, would be given by $F(k) - F(k+1) = -\big(F(k+\Delta k) - F(k)\big)/\Delta k$ (subtract those with degree at least $k+1$ from those with degree at least k). In the continuous version ($\Delta k \to 0$), that probability becomes (see figures on next page):

$$Pr(d_i(t) = k) = -\frac{\partial F}{\partial k} = \frac{2m^2}{k^3} \ ,$$

and we see the emergence of the power law behavior $k^{-\alpha}$ with exponent $\alpha = 3$.
The exponent itself does not depend on m, as will be seen in the programming assignment.


$d_i(t) = m\sqrt{t/t_i}$ for $m=2$, $t=50$
degree
node
fraction with
degree $\geq k$ (for $k=6$)
(goes as m^2/k^2)


degree
node
fraction with
degree $=k$ (for $k=6$)
(goes as $2m^2/k^3$)

Now consider networks whose power law exponent can vary with a parameter p, as introduced in section 18.3 of Easley/Kleinberg chpt.18 (and then predicted in pp. 555–559 to have power law exponent is $\alpha = 1 + 1/(1-p)$ for the degree distribution). This model is more in the spirit of web pages than a social network, with directed links. When a new page (node) j is added, it is given a link (edge) to an earlier page, according to the probabilistic rule: (a) With probability p, page j links to page i chosen uniformly at random from among all earlier pages; (b) With probability $1-p$, page j instead links to a page i chosen with probability proportional to i's current number of in-links. (Rule (b) is along the lines of the previous "preferential attachment" model, and rule (a) is needed to permit discovery of pages that start with zero in-links [via the fully random process]).

Let $x_j(t)$ be the number of in-links to node j at time t (i.e., the in-degree, playing the role of the degree $d_i(t)$ in the previous model). The condition that the node has zero in-coming links when created means that $x_j(j) = 0$. The goal is to determine the expected number of nodes with k in-links at time t. According to the above rules, the probability that a new node created at time $t+1$ links to node j is $p/t + (1-p)x_j(t)/t$ (since at time t, by rule (a) j is chosen from t nodes with uniform probability $1/t$, and by rule (b) the choice is instead according to node j's fraction of in-links, $x_j(t)/t$).

Again approximating with continuous time t, the discrete dynamics becomes

$$\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} = p\frac{1}{t} + (1-p)\frac{x_j(t)}{t} = p\frac{1}{t} + q\frac{x_j(t)}{t}$$

(where $q = 1-p$), with the boundary condition $x_j(t=j) = 0$. Rewriting as $\frac{\mathrm{d}x_j}{p+qx_j(t)} = \frac{\mathrm{d}t}{t}$, this integrates to $\ln(p + qx_j(t)) = q\ln t + c$, or equivalently $p + qx_j(t) = At^q$, where A is a constant determined by the boundary condition. Since $x_j(t) = \frac{1}{q}(At^q - p)$, the condition $0 = x_j(j) = \frac{1}{q}(Aj^q - p)$ implies that $A = p/j^q$, and the solution can be written

$$x_j(t) = \frac{p}{q}\big((t/j)^q - 1\big) \ .$$

In their treatment, E/K keep track of all the terms in the above, but for determining the number of nodes with degree k at (large) time t it only matters that $x_j(t)$ grows in time as $x_j(t) \approx a(t/j)^q$. The fraction of nodes that have $x_j(t) \geq k$ is determined by the condition $a(t/j)^q \geq k$, i.e., the fraction of nodes $F(k)$ of interest consists of the early ones (small j), with $j/t \leq a^{1/q}k^{-1/q}$. With the fraction with *at least* in-degree k behaving as $F(k) \sim k^{-1/q}$, then as before the fraction with exactly k is given by

$$f(k) = -\frac{\mathrm{d}F}{\mathrm{d}k} \sim k^{-1-1/q} \ ,$$

and we see the emergence of the power law with exponent $\alpha = 1 + 1/q = 1 + 1/(1-p)$. The limit $p \to 1$ goes back to the random network, where $\alpha \to \infty$ signals loss of the power law behavior (the tail is extinguished). In the $p \to 0$ limit, the exponent $\alpha \to 2$, and the tail of the distribution is that much more pronounced. The exponent drops to under 3, since unlike in the earlier model, starting the nodes with in-degree zero allows nodes with even larger in-degree to develop via a more skewed probability distribution.