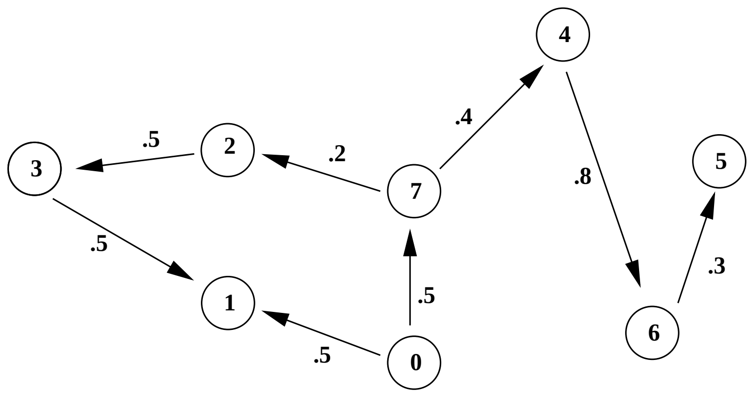

Consider the Markov chain depicted above. The only recurrent states are 1 and 5, each trivially recurrent since it can only go to itself, and hence each forms its own recurrent class. (The stationary distributions have $p_1 = 1$ and $p_5 = 1$.)

Assume the Markov chain is in state 0 just before the first step, some questions:

(a) What is the probability of being in state 7 after n steps?

The unique path is a first step from 0 to 7, then $n-1$ steps from 7 to itself, hence the probability is $.5(.4)^{n-1}$.

(b) What is the probability of reaching state 2 for the first time on the k^{th} step?

The unique path for this is a first step from 0 to 7, then $k-2$ steps from 7 to itself, then a step from 7 to 2, hence the probability is $(.5)(.4)^{k-2}(.2)$

(c) What is the probability of never reaching state 1 ?

The only way to avoid state 1 is to follow the path from $0 \to 7 \to 4$. From state 7 it's twice as likely to go to state 4 as to state 2, therefore 2/3 of the time one will eventually end up in the direction of state 4 from state 7. Since the probability of going from state 0 to state 7 is 1/2, the overall probability of going from state 0 to state 4, and never reaching state 1, is $(1/2) \cdot (2/3) = 1/3$.
[Another way to reproduce the above is to note that the overall probability to reach state 2 from state 7 is to sum over paths that return k times to state 7 (and use $\sum_{n=0}^{\infty} r^n = 1/(1-r)$): $p_{72} \sum_{k=0}^{\infty} p_{77}^k = \frac{p_{72}}{1-p_{77}} = .2/.6 = 1/3$. Similarly, the probability to eventually reach state 4 from state 7 is $p_{74} \sum_{k=0}^{\infty} p_{77}^k = \frac{p_{74}}{1-p_{77}} = .4/.6 = 2/3$.]

(d) What is the expected number of steps until hitting state 5?

If the probability of leaving a state is p, then the probability of leaving after exactly n steps is $(1-p)^{n-1}p$. With $q = 1-p$, the expectation value for the number of steps is

$$E[n] = \sum_{n=0}^{\infty} n\, q^{n-1} p = p \frac{\partial}{\partial q} \sum_{n=0}^{\infty} q^n = p \frac{\partial}{\partial q} \frac{1}{1-q} = p \frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \ .$$

Thus the expected numbers of steps to remain at states 7,4,6 respectively are $1/.6 = 5/3$, $1/.8 = 5/4$, $1/.3 = 10/3$, and the expected number of steps until hitting state 5 is thus $1{+}5/3{+}5/4{+}10/3 = 7\,1/4$. [Note that the number of steps to leave a state is also called the "waiting time". It gives the expected number of times to flip a first H as $1/p = 1/(1/2) = 2$, and the expected number of times to roll a first 1 as $1/p = 1/(1/6) = 6$.]

(e) What is the probability of eventually hitting state 5?

This is the same probability of 1/3 as (c), since any path that doesn't reach 1 eventually hits 5. More specifically, the probability of going from state 0 to state 7 is 1/2, the probability of eventually going from $7 \to 4$ is 2/3, and from state 4 one eventually hits state 5, so the overall probability of going from state 0 to state 5 is $p(0 \to 5) = (1/2){\cdot}(2/3) = 1/3$.

(f) What is the probability of being in state 4 after two steps, given that one is in state 5 after 8 steps?

This is given by the joint probability of being in state 4 after two steps *and* being in state 5 after 8 steps, divided by the overall probability of being in state 5 after 8 steps

$$p(\text{state 4 after 2 steps} \mid \text{state 5 after 8 steps}) = \frac{p(\text{state 4 after 2 steps , state 5 after 8 steps})}{p(\text{state 5 after 8 steps})} \ .$$

The probability of being in state 5 after 8 steps is

$$p(\text{state 5 after 8 steps}) = p_{07}\, p_{74}\, p_{46}\, p_{65} \sum_{\substack{\{i,j,k,\ell\ | \\ i+j+k+\ell=4\}}} p_{77}^i\, p_{44}^j\, p_{66}^k\, p_{55}^\ell = .5{\cdot}.4{\cdot}.8{\cdot}.3{\cdot}(6.6759) \approx .32\ ,$$

where the sum is over all ways of returning to states 7,4,6, and 5 for an overall total of four times. (Note in this case that $p_{55} = 1$ so it doesn't matter when state 5 is reached or how many times one returns to it.) There are many ways to calculate the above sum. One way is to consider the product

$$(1 + .6x + .6^2x^2 + .6^3x^3 + .6^4x^4)(1 + .2x + .2^2x^2 + .2^3x^3 + .2^4x^4)(1 + .7x + .7^2x^2 + .7^3x^3 + .7^4x^4)$$
$$= 1 + 1.5x + 1.57x^2 + 1.419x^3 + 1.1869x^4 + \ldots$$

The powers of x count the sum of the number of steps back to states 7,4,6, so returning to those states four or fewer times corresponds to $1 + 1.5 + 1.57 + 1.419 + 1.1869 = 6.6759$.

The joint probability of being in state 4 after two steps and being in state 5 after 8 steps is given by the same sum as above except with $i = 0$, since there can be no transitions from state 7 back to itself if state 4 is reached in just 2 steps. This gives

$$p(\text{state 4 after 2 steps , state 5 after 8 steps}) =$$
$$= p_{07}\, p_{74}\, p_{46}\, p_{65} \sum_{\{j,k,\ell\ |\ j+k+\ell=4\}} p_{44}^j\, p_{66}^k\, p_{55}^\ell = .5 \cdot .4 \cdot .8 \cdot .3 \cdot (3.3825) \approx .162\ .$$

In this case the sum was given by considering the product

$$(1 + .2x + .2^2x^2 + .2^3x^3 + .2^4x^4)(1 + .7x + .7^2x^2 + .7^3x^3 + .7^4x^4)$$
$$= 1 + .9x + .67x^2 + .477x^3 + .3355x^4 + \ldots$$

where the power of x counts the combined number of steps back to 4 and 6, and for a total of four or fewer times corresponds to the sum $1 + .9 + .67 + .477 + .3355 = 3.3825$.

The result for the conditional probability is

$$p(\text{state 4 after 2 steps} \mid \text{state 5 after 8 steps}) = \frac{.5 \cdot .4 \cdot .8 \cdot .3 \cdot (3.3825)}{.5 \cdot .4 \cdot .8 \cdot .3 \cdot (6.6759)} \approx .506\ .$$