Telegraphic review of exponentials and logarithms:

x^m represents the result of multiplying m factors of x, and satisfies the useful relations

$$x^m x^n = x^{m+n}\ ,\quad (x^n)^m = x^{nm}\ ,\quad x^0 = 1\ . \tag{E1}$$

The exponential function $f(x) = y^x$ is frequently employed with $y = \mathrm{e}$, where $\mathrm{e} = 2.71828182845905...$ is a transcendental number, which can be introduced as follows. Consider investing \$1 at a 100%/year interest rate for a year. If interest is compounded a single time at the end of the year, then the result is $\$1\cdot(1+1) = \$2$. If instead interest is compounded twice, at 50% after a half year and the remaining 50% at the end of the year, then the result is the slightly larger, $\$1\cdot(1+1/2)(1+1/2) = \$2.25$. Now consider breaking up the time interval into N pieces, so that interest is compounded N times, then the total is $(1+1/N)^N$. In the limit that N becomes arbitrarily large (compounded continuously), the result converges to

$$\lim_{N\to\infty}(1+1/N)^N = \mathrm{e}\ . \tag{D1}$$

If the interest rate is instead only 5% per year compounded continuously, then the result would be $\lim_{N\to\infty}(1+.05/N)^N = \lim_{M\to\infty}(1+1/M)^{.05M} = \mathrm{e}^{.05} \approx 1.051271096$. For general interest rate x, it follows that $\lim_{N\to\infty}(1+x/N)^N = \lim_{M\to\infty}(1+1/M)^{Mx} = \mathrm{e}^x$. This function can also be written as an infinite sum by expanding the product, and keeping only the terms that survive in the large N limit:

$$\begin{aligned}\mathrm{e}^x &= \lim_{N\to\infty}\Big(1+\frac{x}{N}\Big)^N = \lim_{N\to\infty}\Big(1+N\frac{x}{N}+\frac{N(N-1)}{2!}\Big(\frac{x}{N}\Big)^2+\cdots+\binom{N}{m}\Big(\frac{x}{N}\Big)^m+\cdots\Big)\\ &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^m}{m!}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}\ .\end{aligned} \tag{E2}$$

The logarithm function is the inverse of the exponential function: if $y^x = X$ then $x = \log_y X$. For example, $10^3 = 1000$ so $\log_{10} 1000 = 3$, and $2^{10} = 1024$ so $\log_2 1024 = 10$.*

Useful properties of the logarithm function follow directly from the corresponding properties $(E1)$ of the exponential function

$$\log XZ = \log X + \log Z\ ,\quad \log X^s = s\log X\ ,\quad \log 1 = 0\ . \tag{L1}$$

(For example if $X = y^x$, $Z = y^z$, then $\log_y XZ = \log_y y^x y^z = x+z = \log_y X + \log_y Z$, and $\log X^s = \log_y y^{xs} = xs = s\log_y X$. Logarithms to different bases y, z are simply related by a numerical factor $\log_z y$:

$$\log_y x = \frac{\log_z x}{\log_z y}$$

* Note that the rough relation $10^3 \approx 2^{10}$ implies that $10^{3/10} \approx 2$ and $2^{10/3} \approx 10$, which permits estimating $\log_{10} 2 \approx 3/10$ and $\log_2 10 \approx 10/3$, good approximations to the actual values of 0.30103... and 3.32192..., respectively.

(following from $\log_z x = \log_z y^{\log_y x} = \log_y x \cdot \log_z y$).

The special notation $\ln = \log_{\mathrm{e}}$ is used for logarithms to the base e. For example, $\ln 2 = \log_{\mathrm{e}} 2 \approx .693$. The series expansion for the function $\ln(1+t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ can be determined by setting $\mathrm{e}^x = 1+t$ (defined so that t is small when x is small) and substituting in $(E2)$:

$$1+t = 1+(a_1t+a_2t^2+a_3t^3+\cdots)+\frac{1}{2}(a_1t+a_2t^2+a_3t^3+\cdots)^2+\frac{1}{3!}(a_1t+a_2t^2+a_3t^3+\cdots)^3+\cdots\ .$$

Comparing the coefficients of powers of t gives $a_1 = 1$, $a_2 + a_1^2/2 = 0$, $a_3 + 2a_1a_2/2 + a_1^3/6 = 0, \ldots$, and we infer that $a_2 = -1/2$, $a_3 = 1/2 - 1/6 = 1/3$, i.e., $\ln(1+t) = t - t^2/2 + t^3/3 - \cdots$. Letting $t \to -t$, the full result can be written in the form

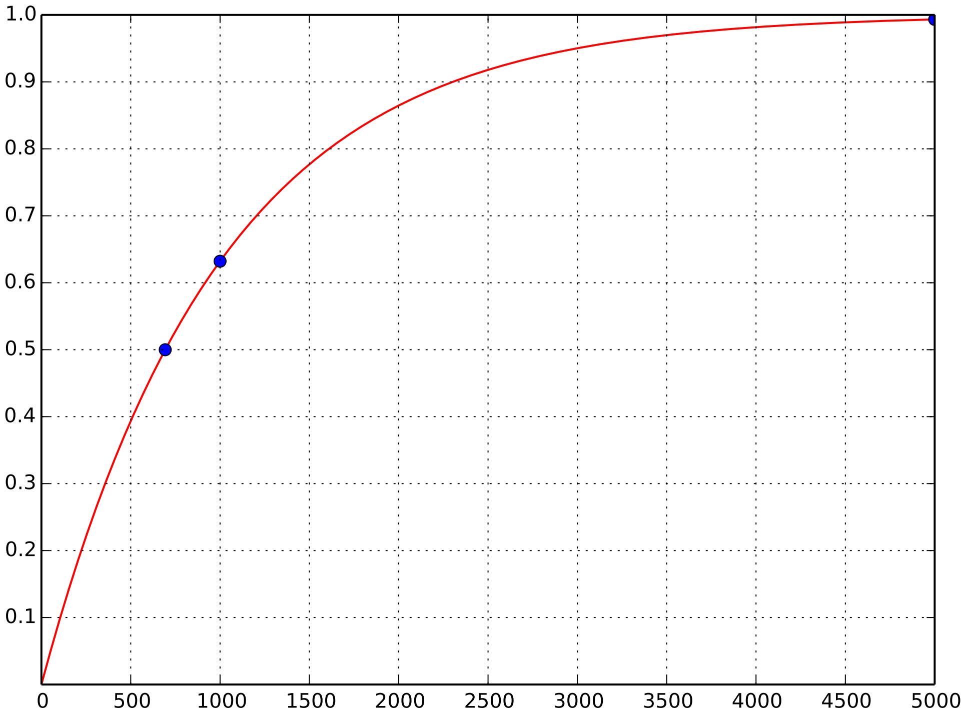
$$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{t^n}{n}\ . \qquad (L2)$$

Example: Suppose a software program has a bug that occurs with probability $p = 1/1000$. a) How many times do we need to run the program in order to have a 50% chance of seeing the bug occur? b) What is the probability of seeing the bug if we run the program 1000 times?

a) The probability of not seeing the problem occur in a single run is $P_1 = 1-p$ so, assuming that successive runs are independent, the probability of seeing no problem after N runs is $P_N = (1-p)^N$. For $(1-p)^N = 1/2$, we find $N\ln(1-p) = \ln(1/2)$, or using $(L2)$ to lowest order, $-Np \approx -\ln 2$, so $N \approx (1/p)\ln 2 = 1000 \ln 2 \approx 693$, so after $N = 693$ trials the probability of no problem falls to roughly 50%.*

b) $P_{1000} = (1 - \frac{1}{1000})^{1000}$. 1000 is large so by $(D1)$ this is approximated by $P_{1000} \approx \mathrm{e}^{-1} \approx .368$. (Equivalently, using $(L2)$ we can estimate $\ln P_{1000} = 1000\ln(1-\frac{1}{1000}) \approx 1000(-\frac{1}{1000}) = -1$, so again $P_{1000} \approx \mathrm{e}^{-1}$). The probability of seeing the bug is therefore $1 - P_{1000} \approx .632$, i.e., has climbed to roughly 63.2% after 1000 trials.

* This problem is directly analogous to the problem of nuclear decay, where p is instead the probability per unit time of decay of some nuclear species. (In the original problem, p was a probability per trial, so a "trial" becomes an infinitesimal time step in the nuclear problem.) The "half-life", i.e., the time for half of some sample to be likely to decay, is given by the equivalent formula $\tau = (1/p)\ln 2$.

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figure(figsize(8,6))
N=arange(5001.)
plot(1-exp(-N/1000),'r')
points = array([693,1000,5000])
plot(points,1-exp(-points/1000.),'o')
xticks(range(0,5001,500))
yticks(arange(.1,1.1,.1))
grid('on')
savefig('eprob.pdf')
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Figure: probability of seeing a problem, given by $1-(999/1000)^N \approx 1-\mathrm{e}^{-N/1000}$. The points plotted are at the values (693,.5), (1000,.632), and (5000,.993), corresponding to 50% at $N=693$, 63.2% at $N=1000$, and 99.3% at $N=5000$.