As one example of the central limit theorem, consider the familiar Bernoulli distribution for n successes in $m+n$ trials:

$$p(n,m\,|\,q)=\frac{(n+m)!}{n!\,m!}q^n(1-q)^m$$

in the limit of large m,n. Recall Stirling's formula (see next page) that for large N, $N!\sim \mathrm{e}^{-N}N^N\sqrt{2\pi N}$, so we can write

$$\frac{(n+m)!}{n!\,m!}q^n(1-q)^m\sim\frac{\sqrt{n+m}}{\sqrt{2\pi nm}}\frac{(n+m)^{n+m}}{n^n m^m}q^n(1-q)^m=\frac{1}{\sigma\sqrt{2\pi}}\Big(\frac{q}{n/(m+n)}\Big)^n\Big(\frac{1-q}{m/(m+n)}\Big)^m\ .$$

To simplify this, we set $p=n/(n+m)$, so that $1-p=m/(n+m)$, and let $\sigma=\sqrt{(n+m)p(1-p)}$, giving

$$p(n,m\,|\,q)\sim\frac{1}{\sigma\sqrt{2\pi}}\Big(\frac{q}{p}\Big)^n\Big(\frac{1-q}{1-p}\Big)^m=\frac{1}{\sigma\sqrt{2\pi}}\ \mathrm{e}^{(n+m)\big(p\ln\frac{q}{p}+(1-p)\ln\frac{1-q}{1-p}\big)}\ . \qquad (1)$$

The function on the right is strongly peaked near its maximum at $p=p_0$, where the exponent $(n+m)f(p)$ can be approximated in terms of the Taylor series $f(p)\approx f(p_0)+f'(p)|_{p=p_0}(p-p_0)+\frac{1}{2}f''(p)|_{p=p_0}(p-p_0)^2$. The point p_0 where the maximum occurs is determined by the vanishing of the first derivative $f'(p)|_{p=p_0}=0$. This derivative satisfies

$$f'(p)=\frac{d}{dp}f(p)=\frac{d}{dp}\Big(p\ln\frac{q}{p}+(1-p)\ln\frac{1-q}{1-p}\Big)=\ln\frac{q}{p}-\ln\frac{1-q}{1-p}\ ,$$

and thus vanishes at $p=q$. We see that the value at the maximum as well vanishes, $f(q)=0$, so the function is approximated by $f(p)\approx\frac{1}{2}f''(p)|_{p=q}(p-q)^2$ near the maximum. Taking the second derivative gives

$$f''(p)=\frac{d}{dp}f'(p)=\frac{d}{dp}\Big(\ln\frac{q}{p}-\ln\frac{1-q}{1-p}\Big)=-\frac{1}{p}-\frac{1}{1-p}=-\frac{1}{p(1-p)}\ .$$

Inserting $f(p)\approx-\frac{(p-q)^2}{2q(1-q)}$ into the exponent of (1) and changing variables from p back to n shows that the Bernoulli form tends to a normal distribution in this large n,m limit:

$$p(n,m\,|\,q)\sim\frac{1}{\sigma\sqrt{2\pi}}\ \mathrm{e}^{-\frac{n+m}{2q(1-q)}(p-q)^2}=\frac{1}{\sigma\sqrt{2\pi}}\ \mathrm{e}^{-\frac{(n-\mu)^2}{2\sigma^2}} \qquad (2)$$

where as expected $\mu=(n+m)q$ and $\sigma=\sqrt{(n+m)q(1-q)}$.

To prove Stirling's formula, $N! \sim \sqrt{2\pi N} N^N \mathrm{e}^{-N}$ for large N, we first recall the method of steepest descent. An integral of the form,

$$I = \int_a^b \mathrm{e}^{Nf(x)} \mathrm{d}x \ ,$$

when N is large, is dominated by the "critical points" x_0 of f, at which $f'(x_0) = 0$. At any such point, we can approximate in the near vicinity of x_0,

$$f(x) \approx f(x_0) - \frac{1}{2}|f''(x_0)|(x - x_0)^2 \ .$$

Assuming only a single critical point, we can then approximate

$$I \approx \mathrm{e}^{Nf(x_0)} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}N|f''(x_0)|x^2} \,\mathrm{d}x = \mathrm{e}^{Nf(x_0)} \sqrt{\frac{2\pi}{N|f''(x_0)|}} \tag{1}$$

(where we have also extended the limits of integration from $-\infty$ to ∞, assuming that only the region near x_0 matters anyway, and then used $\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \sqrt{\pi/a}$ †).

To apply to the factorial function, we first need an integral representation. The function

$$\Gamma(N+1) = \int_0^\infty \mathrm{e}^{-x}\, x^N \mathrm{d}x$$

satisfies the recursion relation

$$\Gamma(N+1) = -\mathrm{e}^{-x}\, x^N \Big|_0^\infty + N \int_0^\infty \mathrm{e}^{-x}\, x^{N-1}\, \mathrm{d}x = N\,\Gamma(N)$$

(using integration by parts). Together with the boundary condition $\Gamma(1) = \int_0^\infty \mathrm{e}^{-x} \mathrm{d}x = 1$, we see that $\Gamma(N+1) = N!$ for integer N.

So we write

$$N! = \Gamma(N+1) = N^{N+1} \int_0^\infty \mathrm{e}^{N(\ln z - z)} \mathrm{d}z$$

where $z = x/N$. Hence $f = \ln z - z$, $f' = 1/z - 1$, $f'' = -1/z^2$, and $z_0 = 1$. Eqn. (1) gives

$$N! \approx N^{N+1}\, \mathrm{e}^{-N} \sqrt{\frac{2\pi}{N}} = \sqrt{2\pi N}\, N^N \mathrm{e}^{-N} \ .$$

† If we write $J = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2}$, then using r, θ cylindrical coordinates its square can be written $J^2 = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-x^2-y^2} = \int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-r^2} = 2\pi\Big(-\frac{1}{2}\mathrm{e}^{-r^2}\Big|_0^\infty\Big) = \pi$. Thus $J = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \sqrt{\pi/a}$.