A *tree* is a connected graph with no cycles. A *forest* is a graph with each connected component a tree. A *leaf* in a tree is any vertex of degree 1.
Example Figure 11 shows a tree and a forest of 2 trees.

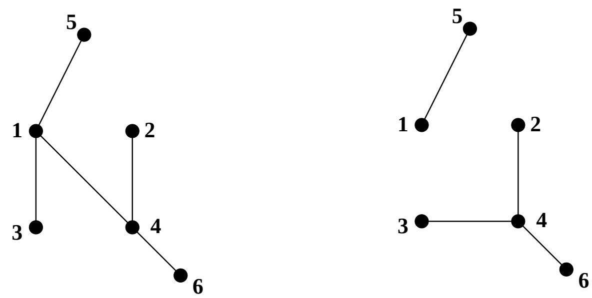

Figure 11: A tree and a forest.

Proposition. *For any tree* $T = (V, E)$ *with* $|V| = n$, $|E| = n - 1$.

Proof. Consider any leaf of T. This vertex is adjacent to exactly one edge. Remove this vertex and edge contributing 1 each to the number of vertices and edges. Continue removing leaf / edge pairs until we are left with just a single edge. A graph with a single edge has one more vertex than edge, hence the total number of edges is one less than the total number of vertices. □

A graph G is *planar* if there exists an embedding of G into the plane such that no two edges cross.
Example: The graph on 4 vertices with edges $(1,2)$ $(2,3)$ $(3,4)$ and $(4,1)$ is planar. Figure 12 shows this graph drawn with 1 edge crossing and with no edge crossings.

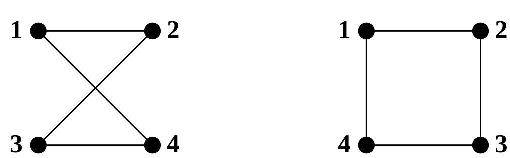

Figure 12: Two representations of the cycle of length 4.

The edges of a planar embedding of a graph divide the plane into regions. Let f be the number of regions of a planar graph, e the number of edges and v the number of vertices.

Theorem. *(Euler's formula) For any connected planar graph, $v - e + f = 2$.*

Proof. We proceed by induction on the number of edges e. Consider the case $e = 1$. There is only one such graph. This graph has $v = 2$, $e = 1$ and $f = 1$. Hence $v - e + f = 2$. Assume the formula holds for any connected planar graph on n edges. Consider a connected planar graph G with $n + 1$ edges, v vertices and f regions. Form G' with statistics e', v', and f' by removing any edge which results in another connected graph. In this case, $v' = v$, $e' = e - 1$, and $f' = f - 1$. (why?) Therefore we have $2 = v' - e' + f' = v - (e - 1) + (f - 1) = v - e + f$.

In forming G', it could have been that removing any edge of G resulted in a disconnected graph. In this case, G is a tree. (why?) Using the proposition above, we know that for any tree $v - e + f = v - (v - 1) + 1 = 2$.

□

Proposition. *For any connected planar graph with $v \geq 3$, $e \leq 3v - 6$.*

Proof. Consider tracing out the boundary of any given region F. Count the number of times we traverse an edge and call this the degree of F. If we traced out every region of G, we would traverse each edge exactly twice. Hence the sum of the degrees of all regions is exactly $2e$. Next note that each region has at least 3 edges on its boundary. Therefore we can conclude that $2e \geq 3f$. Using Euler's formula we get: $2e \geq 3(2 - v + e)$ or $e \leq 3v - 6$.

□

We want to consider two common operations on a graph. The *deletion* of an edge in a graph is removing this edge from the graph. The *contraction* of an edge in a graph deletes the edge and identifies its endpoints to a common vertex. A *minor* of a graph G is any new graph formed from G by a series of deletion and contraction operations.

Example Figure 13 shows the deletion and contraction of the edge $(1, 2)$.

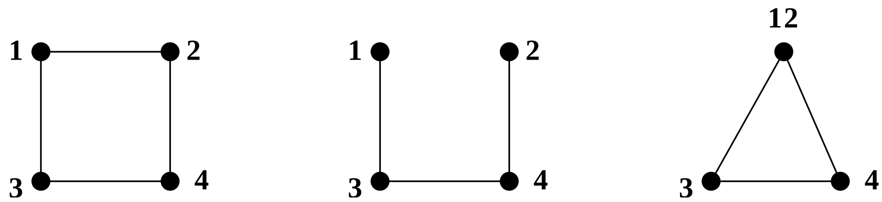

Figure 13: Deletion and Contraction.

Theorem. *A graph is planar iff it does not contain either graph of Figure 14 as a minor.*

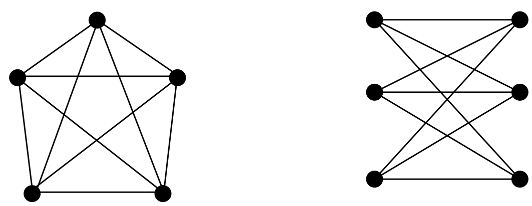

Figure 14: Obstructions to planarity.

A *proper coloring* of a graph is a map f from the vertices of a graph to $\{1, 2, 3, \ldots\}$ such that if $(v_i, v_j) \in E$ then $f(v_i) \neq f(v_j)$. The *chromatic number* of a graph G is the minimum number of colors needed for a coloring of G.

Theorem. *(4-color theorem) The chromatic number of any planar graph is less than or equal to 4.*