We call a vertex v *incident* to an edge e if $v \in e$. The *degree* of a vertex v written $deg(v)$ is the number of edges it is incident to.

Example In G_1, $deg(1) = 4$, $deg(2) = 2$, $deg(3) = 2$, $deg(4) = 4$, $deg(5) = 1$, and $deg(6) = 1$.

In a directed graph, the *outdegree* of a vertex v is the number of edges starting at v and the *indegree* is the number of edges ending at v.

Example In the graph of Figure 3, $outdeg(1) = 2$, $indeg(1) = 1$, $outdeg(2) = 0$, $indeg(2) = 2$, $outdeg(3) = 1$, $indeg(3) = 1$, $outdeg(4) = 2$, $indeg(4) = 2$, $outdeg(5) = 0$, $indeg(5) = 1$, $outdeg(6) = 1$, $indeg(6) = 0$.

A *Eulerian circuit* of a graph G is a cycle which contains each edge of G exactly once.

Theorem. *A connected graph has a Eulerian circuit if and only if for all vertices v, deg(v) is even.*

Proof. Suppose G has a Eulerian circuit. Start at any vertex of G and traverse the circuit. Leaving the initial vertex contributes 1 to its degree, but the circuit will also end at this vertex contributing another 1 to the degree. Anytime we pass through a vertex v_i we enter on one edge and exit at a different edge, this contributes 2 to the deg(v_i). The circuit passes through all edges of the graph, so we are able to compute the degree of each vertex in this way.

Now suppose the deg(v) is even for all v in G. Start at any vertex v_1 of G and form a path through G as long as possible without repeating edges. If this path contains all edges of G we are done. Otherwise, we must have returned to v_1 and traversed all edges incident to it.(why?) We delete the edges of this path and any vertices incident to only edges in this path. Next, start again on the remaining subgraph. This subgraph also has vertices only of even degree. (why?) We repeat the procedure above forming more Eulerian paths. Finally, we glue the paths together to form an Eulerian circuit of G. (how?) $\square$

Example Consider the first graph in Figure 9. First we note that all vertices have even degree. Say we start at vertex 3 and form the following path: $(3, 4)$ $(4, 2)$ $(2, 1)$ $(1, 3)$. Now we can not continue our path. We delete these edges and the vertices 2 and 3. This leaves us with the second graph of Figure 9. Say we start at vertex 1 and form the path $(1, 5)$ $(5, 6)$ $(6, 4)$ $(4, 1)$. Now all edge have been used. We form a Eulerian circuit by gluing these two paths together at a common vertex. In this case we could use 1 or 4. Let us choose vertex 1. This gives the final path: $(3, 4)$ $(4, 2)$ $(2, 1)$ $(1, 5)$ $(5, 6)$ $(6, 4)$ $(4, 1)$ $(1, 3)$.

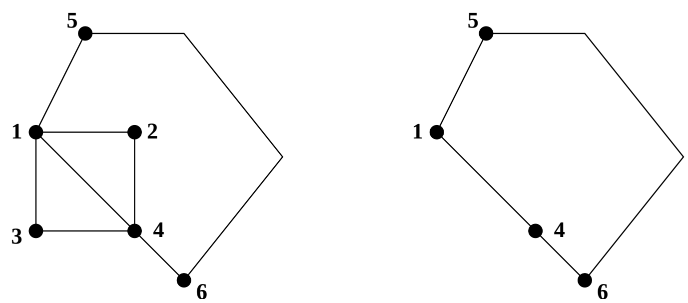

Figure 9: Forming an Eulerian circuit.

A *Hamiltonian circuit* of a graph G is a cycle which passes through each vertex exactly once.

Theorem. *(Dirac's Theorem) A Graph on n vertices has a Hamiltonian circuit if the degree of each vertex is at least $n/2$.*

Proof. We will prove the theorem by contradiction. Suppose we have a graph G such that the degree of each vertex is at least $n/2$, but G does not have a Hamiltonian circuit. First we will form a graph G' by adding certain edges to G. Namely, add all possible edges to G such that the resulting graph has no Hamiltonian circuit. This can only increase the degree of any vertex. Notice that G' has the property that if we add any edge to G' we would form a Hamiltonian circuit. This implies that for any pair of vertices that do not form an edge in G' there does exist a path between them which passes through all other vertices exactly once. (why?) Take two such vertices v_1 and v_n and label the path in order from v_1 to v_n by $\{v_2, v_3, \ldots, v_{n-1}\}$. Consider a vertex v_i such that $(v_1, v_i) \in E$. Then, v_n can not form an edge with v_{i-1}. (why?) Since v_1 is connected to at least $n/2$ other vertices, this leaves less than $n/2$ vertices with which v_n can form an edge. (why?) Hence v_n has degree less than $n/2$ and we have reached a contradiction.

$\square$

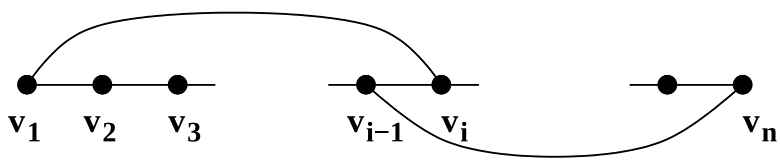

Figure 10: Dirac's Theorem.